Estimates of the Linearization of Circle Diffeomorphisms

Mostapha Benhenda

Estimates of the Linearization of Circle Diffeomorphisms

Mostapha Benhenda*

September 30, 2011

Abstract

A celebrated theorem by Herman and Yoccoz asserts that if the rotation number α of a C^∞-diffeomorphism of the circle f satisfies a Diophantine condition, then f is C^∞-conjugated to a rotation. In this paper, we establish explicit relationships between the C^k norms of this conjugacy and the Diophantine condition on α. To obtain these estimates, we follow a suitably modified version of Yoccoz's proof.

1 Introduction

In his seminal work, M. Herman [5] shows the existence of a set A of Diophantine numbers of full Lebesgue measure such that for any rotation number $\alpha \in A$ of a circle diffeomorphism f of class C^ω (resp. C^∞), there is a C^ω-diffeomorphism (resp. C^∞-diffeomorphism) h such that $hfh^{-1} = R_\alpha$. In the C^∞ case, J. C. Yoccoz [14] extended this result to all Diophantine rotation numbers. Results in analytic class and in finite differentiability class subsequently enriched the global theory of circle diffeomorphisms [11, 9, 8, 13, 7, 15, 4, 10]. In the perturbative theory, KAM theorems usually provide a bound on the norm of the conjugacy that involves the norm of the perturbation and the Diophantine constants of the number α (see [5, 12, 3] for example). We place ourselves in the global setting, we compute a bound on the norms of this conjugacy h in function of k, $|Df|_0$, $W(f)$, $|Sf|_{k-3}$, β and C_d.

To obtain these estimates, we follow a suitably modified version of Yoccoz's proof. Indeed, Yoccoz's proof needs to be modified because a priori, it does not exclude the fact that the set:

$$E_X = \left\{ |Dh|_0 \,/ \exists f \in \mathrm{Diff}_+^k(\mathbb{T}^1),\ f = h^{-1} R_\alpha h, \alpha \in DC(\beta, C_d), \max\left(k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}\right) \leq X \right\}$$

could be unbounded for any fixed $X > 0$.

These estimates have natural applications to the global study of circle diffeomorphisms with Liouville rotation number: in [2], they allow to show the following results: 1) Given a diffeomorphism f of rotation number α, for a Baire-dense set of α, it is possible to accumulate R_α with a sequence $h_n f h_n^{-1}$, h_n being a diffeomorphism. 2) Given two commuting diffeomorphisms f and g, with the rotation number α of f belonging to a specified Baire-dense set, it is possible to approach each of them by commuting diffeomorphisms f_n and g_n that are differentiably conjugated to rotations.

*Laboratoire d'Analyse, Geometrie et Applications, Paris 13 University, 99 Avenue J.B. Clement, 93430 Villetaneuse, France. Contact: mostaphabenhenda@gmail.com

1.1 Notations

We follow the notations of [14].

- The circle is noted $\mathbb{T}^1$. The group of orientation-preserving circle diffeomorphisms of class C^r is denoted $\mathrm{Diff}^r_+(\mathbb{T}^1)$. The group of $\mathbb{Z}$-periodic diffeomorphisms of class C^r of the real line is noted $C^r(\mathbb{T}^1)$. We often work in the universal cover $D^r(\mathbb{T}^1)$, which is the group of diffeomorphisms f of class C^r of the real line such that $f - Id \in C^r(\mathbb{T}^1)$. Note that if $f \in D^r(\mathbb{T}^1)$ and $r \geq 1$, then $Df \in C^{r-1}(\mathbb{T}^1)$.
- The derivative of $f \in D^1(\mathbb{T}^1)$ is noted Df. The Schwartzian derivative Sf of $f \in D^3(\mathbb{T}^1)$ is defined by:

$$Sf = D^2 \log Df - \frac{1}{2}(D \log Df)^2$$

- The total variation of the logarithm of the first derivative of f is:

$$W(f) = \sup_{a_0 \leq \ldots \leq a_n} \sum_{i=0}^{n} |\log Df(a_{i+1}) - \log Df(a_i)|$$

- For any continuous and $\mathbb{Z}$-periodic function ϕ, let:

$$|\phi|_0 = \|\phi\|_0 = \sup_{x \in \mathbb{R}} |\phi(x)|$$

- Let $0 < \gamma' < 1$. $\phi \in C^0(\mathbb{T}^1)$ is Holder of order γ' if:

$$|\phi|_{\gamma'} = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^{\gamma'}} < +\infty$$

  Let $\gamma \geq 1$ be a real number. In all the paper, we write $\gamma = r + \gamma'$ with $r \in \mathbb{N}$ and $0 \leq \gamma' < 1$.
- A function $\phi \in C^r(\mathbb{T}^1)$ is said to be of class C^γ if $D^r\phi \in C^{\gamma'}(\mathbb{T}^1)$. The space of these functions is noted $C^\gamma(\mathbb{T}^1)$ and is given the norm:

$$\|\phi\|_\gamma = \max\left(\max_{0 \leq j \leq r} \|D^j\phi\|_0, |D^r\phi|_{\gamma'}\right)$$

  If $\gamma = 0$ or $\gamma \geq 1$, the C^γ-norm of ϕ is indifferently denoted $\|\phi\|_\gamma$ or $|\phi|_\gamma$. Thus, when possible, we favor the simpler notation $|\phi|_\gamma$.
- For $\alpha \in \mathbb{R}$ (respectively, $\alpha \in \mathbb{T}^1$), we denote $R_\alpha \in D^\infty(\mathbb{T}^1)$ (respectively, $R_\alpha \in \mathrm{Diff}^\infty_+(\mathbb{T}^1)$). the map $x \mapsto x + \alpha$.
- An irrational number $\alpha \in DC(C_d, \beta)$ satisfies a Diophantine condition of order $\beta \geq 0$ and constant $C_d > 0$ if for any rational number p/q, we have:

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C_d}{q^{2+\beta}}$$

  Moreover, if $\beta = 0$, then α is of constant type C_d.

- Let $\alpha_{-2} = \alpha$, $\alpha_{-1} = 1$. For $n \geq 0$, we define a real number α_n (the *Gauss sequence* of α) and an integer a_n by the relations $0 < \alpha_n < \alpha_{n-1}$ and

$$\alpha_{n-2} = a_n \alpha_{n-1} + \alpha_n$$

- In the following statements, $C_i[a, b, ...]$ denotes a positive numerical function of real variables $a, b, ...$, with an explicit formula that we compute.

  $C[a, b, ...]$ denotes a numerical function of $a, b, ...$, with an explicit formula that we do not compute.

- We use the notations $a \wedge b = a^b$, $e^{(n)} \wedge x$ the n^{th}- iterate of $x \mapsto \exp x$, $\lfloor x \rfloor$ for the largest integer such that $\lfloor x \rfloor \leq x$, and $\lceil x \rceil$ for the smallest integer such that $\lceil x \rceil \geq x$.

We recall Yoccoz's theorem [14]:

Theorem 1.1. *Let $k \geq 3$ an integer and $f \in D^k(\mathbb{T}^1)$. We suppose that the rotation number α of f is Diophantine of order β. If $k > 2\beta + 1$, there exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α. Moreover, for any $\eta > 0$, h is of class $C^{k-1-\beta-\eta}$.*

1.2 Statement of the results

1.2.1 C^1 estimations

Theorem 1.2. *Let $f \in D^3(\mathbb{T}^1)$ of rotation number α, such that α is of constant type C_d. There exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α, which satisfies the estimation:*

$$|Dh|_0 \leq e \wedge \left(\frac{C_1[W(f), |Sf|_0]}{C_d} \right)$$

The expression of C_1 is given page 10.

More generally, for a Diophantine rotation number $\alpha \in DC(C_d, \beta)$, we have:

Theorem 1.3. *Let $k \geq 3$ be an integer and $f \in D^k(\mathbb{T}^1)$. Let $\alpha \in DC(C_d, \beta)$ be the rotation number of f. If $k > 2\beta + 1$, there exists a diffeomorphism $h \in D^1(\mathbb{T}^1)$ conjugating f to R_α, which satisfies the estimation:*

$$|Dh|_0 \leq C_2[k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}] \tag{1}$$

The expression of C_2 is given page 23.

Moreover, if $k \geq 3\beta + 9/2$, we have:

$$|Dh|_0 \leq e^{(3)} \wedge \Big(C_3[\beta] C_4[C_d] C_5[|Df|_0, W(f), |Sf|_0] C_6[|Sf|_{\lceil 3\beta + 3/2 \rceil}] \Big) \tag{2}$$

The expressions of C_3, C_4, C_5, C_6 are given page 28.

Let $\delta = k - 2\beta - 1$. When $\delta \to 0$, we have:

$$|Dh|_0 \leq e^{(3)} \wedge \left(\frac{1}{\delta^2} C_7[k, C_d, |Df|_0, W(f), |Sf|_0] + \frac{C[\delta]}{\delta^2} C[k, C_d, |Df|_0, W(f), |Sf|_0, |Sf|_{k-3}] \right) \tag{3}$$

where $C[\delta] \to_{\delta \to 0} 0$. The expression of C_7 is given page 30.

Remark 1.4. Katznelson and Ornstein [9] showed that the assumption $k > 2\beta + 1$ in Yoccoz's theorem is not optimal (instead it is $k > \beta + 2$). Therefore, the divergence of the bound given by estimation (3) is because we compute the bound of the conjugacy by following the Herman-Yoccoz method.

Remark 1.5. Let α_n be the Gauss sequence associated with α. Yoccoz's proof already gives the following result: if $k \geq 3\beta + 9/2$ and if, for any $n \geq 0$,

$$\frac{\alpha_{n+1}}{\alpha_n} \geq C_8[n, k, W(f), |Sf|_{k-3}] \tag{4}$$

then:

$$|Dh|_0 \leq \exp\left(C_9[k, W(f), |Sf|_{k-3}]^{C_{10}(\beta)}\right)|Df|_0^2$$

The expressions of C_8, C_9, C_{10} are given page 30.

1.2.2 C^u estimations

Theorem 1.6. *Let $k \geq 3$ an integer, $\eta > 0$ and $f \in D^k(\mathbb{T}^1)$. Let $\alpha \in DC(C_d,\beta)$ be the rotation number of f. If $k > 2\beta + 1$, there exists a diffeomorphism $h \in D^{k-1-\beta-\eta}(\mathbb{T}^1)$ conjugating f to R_α, which satisfies the estimation:*

$$\|Dh\|_{k-2-\beta-\eta} \leq e^{\left(\lceil \log((k-2-\beta)/\eta)/\log(1+1/(2\beta+3))\rceil\right)} \wedge (C_{11}[\eta, k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3}]) \tag{5}$$

The expression of C_{11} is given page 48.

Moreover, if $k \geq 3\beta + 9/2$, we have:

$$\|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq e \wedge \left(C_{12}[k]e^{(2)} \wedge (2 + C_3[\beta]C_4[C_d]C_5[|Df|_0, W(f), |Sf|_0]C_6[|Sf|_{k-3}])\right) \tag{6}$$

The expression of C_{12} is given page 46.

If α is of constant type, for any $k > 3$, we have:

$$\|Dh\|_{\frac{k}{4}-\frac{1}{2}} \leq e \wedge \left(C_{13}[k]\left[C_{14}[W(f), |Sf|_{k-3}] + \frac{C_1[W(f), |Sf|_0]}{C_d}\right]^4\right) \tag{7}$$

The expressions of C_{13} and C_{14} are given page 47.

2 Preliminaries

Let $f \in D^0(\mathbb{T}^1)$ be a homeomorphism and $x \in \mathbb{R}$. When n tends towards infinity, $(f^n(x) - x)/n$ admits a limit independent of x, noted $\rho(f)$. We call it the *translation number* of f. Two lifts of $f \in \mathrm{Diff}^0_+(\mathbb{T}^1)$ only differ by a constant integer, so this is also the case for their translation numbers. We call the class of $\rho(f)$ mod $\mathbb{Z}$ the *rotation number* of f. We still denote it $\rho(f)$. It is invariant by conjugacy. Let $f \in D^2(\mathbb{T}^1)$. When $\alpha = \rho(f)$ is irrational, Denjoy showed that f is topologically conjugated to R_α. However, this conjugacy is not always differentiable (see [1, 5, 7, 15]). The regularity of this conjugacy depends on the Diophantine properties of the rotation number α (see Yoccoz's theorem 1.1).

Let α be an irrational number. Let the distance of α to the closest integer be:

$$||\alpha|| = \inf_{p \in \mathbb{Z}} |\alpha - p|$$

For $n \geq 1$, $a_n \geq 1$. Let $\alpha = a_0 + 1/(a_1 + 1/(a_2 + ...))$ be the development of α in continued fraction. We denote it $\alpha = [a_0, a_1, a_2, ...]$. Let $p_{-2} = q_{-1} = 0$, $p_{-1} = q_{-2} = 1$. For $n \geq 0$, let p_n and q_n be:

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n q_{n-1} + q_{n-2}$$

We have $q_0 = 1$, $q_n \geq 1$ for $n \geq 1$. The rationals p_n/q_n are called the convergents of α. They satisfy the following properties:

1. $\alpha_n = (-1)^n (q_n \alpha - p_n)$
2. $\alpha_n = ||q_n \alpha||$, for $n \geq 1$
3. $1/(q_{n+1} + q_n) < \alpha_n < 1/q_{n+1}$ for $n \geq 0$.
4. $\alpha_{n+2} < \frac{1}{2}\alpha_n$, $q_{n+2} \geq 2q_n$, for $n \geq -1$

We recall that $DC(C_d, \beta)$ denotes the set of Diophantine numbers of constants β and C_d. One of the following relations characterizes $DC(C_d, \beta)$:

1. $|\alpha - p_n/q_n| > C_d/q_n^{2+\beta}$ for any $n \geq 0$
2. $a_{n+1} < \frac{1}{C_d} q_n^{\beta}$ for any $n \geq 0$
3. $q_{n+1} < \frac{1}{C_d} q_n^{1+\beta}$ for any $n \geq 0$
4. $\alpha_{n+1} > C_d \alpha_n^{1+\beta}$ for any $n \geq 0$

In all the paper, we denote $C'_d = 1/C_d$.

- Let $m_n(x) = f^{q_n}(x) - x$, $n \geq 1$, $x \in \mathbb{T}^1$, let $M_n = \sup_{x \in \mathbb{T}^1} |f^{q_n}(x) - x|$ and $m_n = \inf_{x \in \mathbb{T}^1} |f^{q_n}(x) - x|$.

- For any $\phi, \psi \in C^\gamma(\mathbb{T}^1)$, we have:

$$|\phi\psi|_\gamma \leq ||\phi||_0 |\psi|_\gamma + |\phi|_\gamma ||\psi||_0 \tag{8}$$

$$||\phi\psi||_\gamma \leq ||\phi||_0 |\psi|_\gamma + ||\phi||_\gamma ||\psi||_0 \tag{9}$$

- For any real numbers a and b, $a \vee b$ denotes $\max(a, b)$.

In the rest of the paper, for any integer i, C_i^f denotes a constant depending only on $W(f)$ and $|Sf|_0$ (i.e. C_i^f is a numerical function of these variables). $C_i^{f,k}$ denotes a constant depending only on k, $W(f)$, $|Sf|_0$ and $|Sf|_{k-3}$. C_i denotes a constant that might depend on k, $W(f)$, $|Sf|_0$, $|Sf|_{k-3}$ and also β and C_d.

3 C^1 estimations: constant type

3.1 A 2-parameters family of homographies

In this subsection, we show the existence of a lower bound on the norm of the conjugacy in function of C_d in the particular case of a 2-parameters family of homographies. We also establish an upper bound on the C^1 norm of the conjugacy for this family. These bounds are similar to what is given by the local KAM theory. However, these bounds are very specific to this setting. Our general bounds given in theorems 1.2 and 1.3 are much larger.

Proposition 3.1. *Let $f : \{z \in \mathbb{C}/|z| = 1\} \to \{z \in \mathbb{C}/|z| = 1\}$ defined by $f(z) = h^{-1}R_\theta h(z)$, with $R_\theta(z) = e^{i\theta}z$ and h is a homography defined by:*

$$h(z) = \frac{z-a}{az-1}$$

Let $2 > a > 1$, let C_d such that $C_d^{-1} \geq 6$ is a positive integer; and $0 < \theta = 2\pi C_d \leq \pi/3$, (therefore, $\theta/(2\pi) = [0, C_d^{-1}, 1]$ is of constant type C_d). Let $\tilde{f} : \mathbb{T}^1 \to \mathbb{T}^1$ the circle diffeomorphism induced by f and $\tilde{h}$ the conjugacy induced by h. We have the following estimation:

$$\frac{2}{\pi} C_{15}(|D\tilde{f}(0)|, |D^2\tilde{f}(0)|)/C_d \leq |D\tilde{h}|_0 \leq 9C_{15}(|D\tilde{f}(0)|, |D^2\tilde{f}(0)|)/C_d$$

Proof. For any $\phi \in \mathbb{R}/\mathbb{Z}$, we can write $h(e^{i\phi}) = e^{i\tilde{h}(\phi)}$. By differentiating this expression, we have:

$$D\tilde{h}(\phi) = e^{i\phi}\frac{Dh(e^{i\phi})}{h(e^{i\phi})}$$

and

$$Dh(z) = \frac{(a-1)(a+1)}{(az-1)^2}$$

Therefore

$$D\tilde{h}(\phi) = e^{i\phi}\frac{a^2-1}{(ae^{i\phi}-1)(e^{i\phi}-a)}$$

$|D\tilde{h}(\phi)|$ reaches its maximum for $\phi = 0$, and $|D\tilde{h}|_0 = \frac{a+1}{a-1}$.
Moreover, we have:

$$\frac{D^2\tilde{f}(\phi)}{D\tilde{f}(\phi)} = i + ie^{i\phi}\frac{D^2f(e^{i\phi})}{Df(e^{i\phi})} - ie^{i\phi}\frac{Df(e^{i\phi})}{f(e^{i\phi})}$$

Since $\frac{D^2\tilde{f}(\phi)}{D\tilde{f}(\phi)} \in \mathbb{R}$ and $D\tilde{f}(\phi) \in \mathbb{R}$, we have:

$$\left|\frac{D^2f(1)}{Df(1)}\right| = \left(\left(\frac{D^2\tilde{f}(0)}{D\tilde{f}(0)}\right)^2 + \left(D\tilde{f}(0)-1\right)^2\right)^{1/2} = 8C_{15}(|D\tilde{f}(0)|, |D^2\tilde{f}(0)|)$$

Therefore, in order to get the proposition, it suffices to show:

$$\frac{1}{4\pi C_d}\left|\frac{D^2f(1)}{Df(1)}\right| \leq |Dh|_0 \leq \frac{9}{8C_d}\left|\frac{D^2f(1)}{Df(1)}\right|$$

Let us write

$$f(z) = \frac{(e^{i\theta} - a^2)z - a(e^{i\theta} - 1)}{a(e^{i\theta} - 1)z - (a^2 e^{i\theta} - 1)} = \frac{bz - c}{cz + d}$$

We have

$$Df(z) = \frac{db + c^2}{(cz + d)^2}$$

and

$$D^2 f(z) = -2\frac{Df(z)}{z + d/c}$$

Moreover,

$$Df(1) = \frac{(1 + a)^2 e^{i\theta}}{(ae^{i\theta} + 1)^2}$$

and

$$D^2 f(1) = -2Df(1)\frac{a(e^{i\theta} - 1)}{(1 - a)(1 + ae^{i\theta})}$$

We have

$$\left|\frac{D^2 f(1)}{Df(1)}\right| = 2\left|\frac{a}{1 + ae^{i\theta}}\right| \frac{|e^{i\theta} - 1|}{a - 1}$$

Since $|e^{i\theta} - 1| \geq \sin\theta \geq \frac{2}{\pi}\theta$ (because $0 \leq \theta \leq \pi/2$), then:

$$\left|\frac{D^2 f(1)}{Df(1)}\right| \geq \frac{4}{\pi}\frac{a}{1 + a}\frac{\theta}{a - 1}$$

Therefore,

$$\left|\frac{D^2 f(1)}{Df(1)}\right| \geq \frac{4}{\pi}\frac{a}{(1 + a)^2}|Dh|_0\theta$$

i.e.

$$\frac{9\pi}{4\theta}\left|\frac{D^2 f(1)}{Df(1)}\right| \geq |Dh|_0$$

Hence the first part of the inequality.

On the other hand, since $\theta \leq \pi/3$, then $|1 + ae^{i\theta}| \geq 1 + a\cos\theta \geq 1 + a/2 \geq \frac{1}{2}(a + 1)$. Furthermore, $|e^{i\theta} - 1|^2 = 2 - 2\cos\theta = 2 - 2(\cos^2\theta/2 - \sin^2\theta/2) = 4\sin^2\theta/2 \leq \theta^2$. Therefore,

$$\left|\frac{D^2 f(1)}{Df(1)}\right| \leq \frac{\theta}{a - 1}\frac{4a}{a + 1} \leq \frac{4\theta}{a - 1} = \frac{4\theta|Dh|_0}{a + 1}$$

i.e.

$$\frac{1}{2}\left|\frac{D^2 f(1)}{\theta Df(1)}\right| \leq |Dh|_0$$

Hence the second part of the inequality.

□

3.2 Proof of theorem 1.2

The proof of theorem 1.2 is divided in three steps. The first step is based on the improved Denjoy inequality, which estimates the C^0-norm of $\log Df^{q_l}$. In the second step, we extend this estimation to $\log Df^N$ for any integer N. To do this, following Denjoy and Herman, we write $N = \sum_{s=0}^{S} b_s q_s$, with b_s integers satisfying $0 \leq b_s \leq q_{s+1}/q_s$ and we apply the chain rule. In the third step, we derive a C^0-estimation of the derivative Dh of the conjugacy h.

The first step is based on the Denjoy inequality:

Proposition 3.2. *Let $f \in \mathit{Diff}_+^3(\mathbb{T}^1)$ and $x \in \mathbb{T}^1$. We have:*

$$|\log Df^{q_l}(x)| \leq W(f)$$

Proposition 3.2 is used to obtain an improved version of Denjoy inequality [14, p.342]:

Lemma 3.3. *Let $f \in \mathit{Diff}_+^3(\mathbb{T}^1)$. We have:*

$$|\log Df^{q_l}|_0 \leq C_{16}^f M_l^{1/2}$$

$$|Df^{q_l} - 1|_0 \leq C_{17}^f M_l^{1/2}$$

Moreover, we can take:

$$C_{16}^f = 2\sqrt{2}(2e^{W(f)} + 1)e^{W(f)}(|Sf|_0)^{1/2}$$

and

$$C_{17}^f = 6\sqrt{2}e^{3W(f)}|Sf|_0^{1/2}$$

In the second step, we estimate $D\log Df^N$ independently of N. This step is based on the following lemma:

Lemma 3.4. *Let $f \in \mathit{Diff}_+^3(\mathbb{T}^1)$ and $M_l = \sup_{x\in\mathbb{T}^1} |f^{q_l}(x) - x|$. We have:*

$$\sum_{l\geq 0} \sqrt{M_l} \leq \frac{1}{\sqrt{C_{19}^f} - C_{19}^f}$$

with

$$C_{19}^f = \frac{1}{\sqrt{1 + e^{-C_{22}^f}}} \tag{10}$$

and:

$$C_{22}^f = 6\sqrt{2}e^{2W(f)}\left(|Sf|_0^{1/2} \vee 1\right) \tag{11}$$

Proof. To obtain this lemma, we need the claim:

Claim 3.5. *Let $f \in Diff^2_+(\mathbb{T}^1)$ of rotation number α, and let p_n/q_n be the convergents of α. Then for all $x \in \mathbb{T}^1$, we have:*

$$[x, f^{2q_{l+2}}(x)] \subset [x, f^{q_l}(x)]$$

Proof. By topological conjugation, it suffices to examine the case of a rotation of angle α. It is also sufficient to take $x = 0$.

By absurd, if the lemma was false, then we would have the following cyclic order on $\mathbb{T}^1$: $-q_{l+2}\alpha \leq (q_{l+2} - q_l)\alpha \leq 0 \leq (q_l - q_{l+2})\alpha \leq q_{l+2}\alpha$. In particular, $(q_{l+2} - q_l)\alpha$ would be closer to 0 than $q_{l+2}\alpha$, which would contradict the fact that

$\|q_{l+2}\alpha\| = \inf\{\|q\alpha\|/0 < q \leq q_{l+2}\}$.

□

For any interval I of the circle, if $|I|$ denotes the length of I, lemma 3.3 implies the estimation:

$$\frac{|f^{q_{l+2}}(I)|}{|I|} \geq e^{-C^f_{22} M^{1/2}_{l+2}}$$

Let $x \in \mathbb{T}^1$ such that $M_{l+2} = f^{q_{l+2}}(x) - x$ and let $I = [x, f^{q_{l+2}}(x)]$. The former estimation implies

$$|f^{2q_{l+2}}(x) - f^{q_{l+2}}(x)| \geq e^{-C^f_{22} M^{1/2}_{l+2}} M_{l+2}$$

By applying claim 3.5, and since $M_n \leq 1$, we obtain:

$$M_{n+2} + e^{-C^f_{22}} M_{n+2} \leq M_{n+2} + e^{-C^f_{22} M^{1/2}_{n+2}} M_{n+2} \leq M_n$$

Therefore, for any $l \geq 0$,

$$M_l \leq (C^f_{19})^{l-1} \tag{12}$$

with

$$C^f_{19} = \frac{1}{\sqrt{1 + e^{-C^f_{22}}}}$$

Estimation (12) above gives:

$$\sum_{l \geq 0} \sqrt{M_l} \leq \frac{1}{\sqrt{C^f_{19}}} \frac{1}{1 - \sqrt{C^f_{19}}} \leq \frac{1}{\sqrt{C^f_{19}} - C^f_{19}}$$

Hence lemma 3.4.

□

Now, let N be an integer. Following Denjoy, since α is of constant type, we can write $N = \sum_{l=0}^{s} b_l q_l$, with b_l integers satisfying $0 \leq b_l \leq q_{l+1}/q_l \leq C_d^{-1}$. By the chain rule and by lemma 3.3, since for all $y \in \mathbb{T}^1$, $Df^N(y) > 0$, then :

$$\begin{aligned}|\log D(f^N)(y)| &= |\log D(f^{\sum_{l=0}^{s} b_l q_l})(y)| = |\textstyle\sum_{l=0}^{s} \sum_{i=0}^{b_s} \log Df^{q_l} \circ f^{iq_l}(y)| \\ &\leq \sup_{0 \leq l \leq s} b_l \textstyle\sum_{l=0}^{s} |\log |D(f^{q_l})|_0| \leq C_d^{-1} C^f_{22} \sum_{l \geq 0} M_l^{1/2}\end{aligned}$$

By taking the upper bound on $y \in \mathbb{T}^1$ and $N \geq 0$, we obtain an estimation of $\sup_{N\geq 0} |\log D(f^N)|$.

We turn to the third step: we relate the norms of Dh and Df^N. By [14], h is C^1 and conjugates f to a rotation. Therefore, we have:

$$\log Dh - \log Dh \circ f = \log Df$$

hence, for all n integer:

$$\log Dh - \log Dh \circ f^n = \log D(f^n)$$

Since there is a point z in the circle such that $Dh(z) = 1$, we then have:

$$|\log Dh \circ f^n(z)| = |\log D(f^n)(z)| \leq \sup_{i\geq 0} |\log D(f^i)|_0$$

Moreover, since $(f^n(z))_{n\geq 0}$ is dense in the circle, and since Dh is continuous, then we obtain:

$$|\log Dh|_0 \leq \sup_{i\geq 0} |\log D(f^i)|_0$$

We conclude:

$$|Dh|_0 \leq \exp\left(C_d^{-1} C_{22}^f \sqrt{e^{C_{22}^f \max(M_0^{1/2}, M_1^{1/2})} + 1}(\sqrt{M_0} + \sqrt{M_1})\right) \tag{13}$$

Finally, since $\max(M_0^{1/2}, M_1^{1/2}) \leq 1$, we obtain:

$$|Dh|_0 \leq \exp\left(C_1^f / C_d\right)$$

where $C_1^f = 2C_{22}^f \sqrt{e^{C_{22}^f} + 1}$. We recall that:

$$C_{22}^f = 6\sqrt{2} e^{2W(f)} \left(|Sf|_0^{1/2} \vee 1\right)$$

Hence the theorem.

Corollary 3.6. *Since* $\frac{1}{\min_{\mathbb{T}^1} Dh} \leq \exp\left(\sup_{i\geq 0} |\log D(f^i)|_0\right)$*, the proof above also provides an estimation on* $\frac{1}{\min_{\mathbb{T}^1} Dh}$*:*

$$\frac{1}{\min_{\mathbb{T}^1} Dh} \leq \exp\left(C_1^f / C_d\right)$$

4 C^1 estimations: non-constant type

We have $\max_{n\geq 0} |Df^n|_0 \leq \max_{n\geq 0} M_n / m_n$, by [14, p. 348]. Therefore, in order to prove theorem 1.3, we can estimate M_n / m_n. To that end, we proceed in two steps: first, we establish some preliminary results. The most important result is corollary 4.6, which gives an estimation of M_{n+1}/M_n in function of M_n, α_{n+1}/α_n and a constant $C_{28}^{f,k}$. This estimation is already given in [14, p. 345], but we still recall the steps to reach it, because we need to estimate the constant $C_{28}^{f,k}$ in function of k, $W(f)$, $|Sf|_0$ and $|Sf|_{k-3}$.

In the second step, we establish an estimation of the C^1-conjugacy, based on a modification of the proof given in [14]. The main idea is to establish an alternative between two possible situations for the sequences M_n and α_n: the "favorable" situation (R_n) and the "unfavorable" situation (R'_n) (proposition 4.10). The "unfavorable" situation only occurs a finite number of times, due to the Diophantine condition on α (propositions 4.12 and 4.14).

In the "favorable" situation (R_n), we can estimate M_{n+1}/α_{n+1} in function of M_n/α_n (see estimation (26)) and likewise, we can estimate α_{n+1}/m_{n+1} in function of α_n/m_n. Therefore, we can estimate M_n/m_n in function of M_{n_4}/m_{n_4}, where n_4 is the integer such that for any $n \geq n_4$, the favorable case occurs (see proposition 4.19). We relate M_{n_4}/m_{n_4} to $Df|_0^{\frac{2}{\alpha_{n_4}}}$ (proposition 4.17), and we compute a bound on α_{n_4} (proposition 4.15). Yoccoz's proof needs to be modified because in its original version, it does not allow to compute a bound on α_{n_4}.

4.1 Preliminary results

First, we recall the following lemmas, which are in [14] (lemmas 3,4 and 5):

Lemma 4.1. *For $l \geq 1$ and $x \in \mathbb{T}^1$, we have:*

$$\sum_{i=0}^{q_{n+1}-1} \left(Df^i(x)\right)^l \leq C_{23}^f \frac{M_n^{l-1}}{m_n(x)^l}$$

with $C_{23}^f(l) = e^{lW(f)}$.

Remark 4.2. This lemma is obtained by applying Denjoy inequality.

Lemma 4.3. *Let $f \in Diff_+^k(\mathbb{T}^1)$, $k \geq 3$. For any $x \in \mathbb{T}^1$, any $n \in \mathbb{N}$, any $0 \leq p \leq q_{n+1}$, we have:*

$$|Sf^p|_0 \leq C_{24}^f \frac{M_n}{m_n^2}$$

$$|Sf^p(x)| \leq C_{24}^f \frac{M_n}{m_n(x)^2}$$

$$|D\log Df^p|_0 \leq C_{25}^f \frac{M_n^{1/2}}{m_n}$$

$$|D\log Df^p(x)| \leq C_{26}^f \frac{M_n^{1/2}}{m_n(x)}$$

with:

- $C_{24}^f = |Sf|_0 e^{2W(f)}$
- $C_{25}^f = \sqrt{2|Sf|_0} e^{W(f)}$
- $C_{26}^f = 9\sqrt{2|Sf|_0} e^{4W(f)}$

Lemma 4.4. *For* $1 \le r \le k-1$, $n \ge 0$, $0 \le p \le q_{n+1}$, $x \in \mathbb{T}^1$, *we have:*

$$|D^r \log Df^p(x)| \le C_{27}^f(r)\left[\frac{M_n^{1/2}}{m_n(x)}\right]^r \tag{14}$$

with

$$C_{27}^f(1) = C_{26}^f,\ C_{27}^f(2) = 82|Sf|_0 e^{8W(f)}$$

and, for $r \ge 3$*:*

$$C_{27}^f(r) = \left[82(2r)^{2r}(1 \vee |Sf|_{r-2})^2 e^{(r+8)W(f)}\right]^{r!}$$

In particular,

$$C_{27}^{f,k} := C_{27}^f(k-1) \le \left[100(2k-2)^{2k-2}(1 \vee |Sf|_{k-3})^2 e^{(k+7)W(f)}\right]^{(k-1)!}$$

Proof of lemma 4.4. The proof follows the line of [14], lemma 5: see appendix 6.1. □

The important preliminary result, corollary 4.6, is obtained from the following proposition. It is obtained by computing the constants in proposition 2 of [14]:

Proposition 4.5. *Let*

$$C_{28}^{f,k} = (k+3)^{(k+3)!} e^{(k+2)!W(f)} (\max(1, |Sf|_{k-3}))^{k!} \tag{15}$$

For any $x \in \mathbb{T}^1$, *we have:*

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \le C_{28}^{f,k}\left[M_n^{(k-1)/2} m_n(x) + M_n^{1/2} m_{n+1}(x)\right] \tag{16}$$

Corollary 4.6.

$$M_{n+1} \le M_n \frac{\frac{\alpha_{n+1}}{\alpha_n} + C_{28}^{f,k} M_n^{(k-1)/2}}{1 - C_{28}^{f,k} M_n^{1/2}} \tag{17}$$

$$m_{n+1} \ge m_n \frac{\frac{\alpha_{n+1}}{\alpha_n} - C_{28}^{f,k} M_n^{(k-1)/2}}{1 + C_{28}^{f,k} M_n^{1/2}}$$

The proof of proposition 4.5 combines the following three lemmas [14, pp. 343-344] (lemmas 6, 7 and 8):

Lemma 4.7. *For any* $x \in \mathbb{T}^1$, *there exists* $y \in [x, f^{q_n}(x)], z \in [f^{q_{n+1}}(x), x]$ *such that*

$$m_{n+1}(y) = \frac{\alpha_{n+1}}{\alpha_n} m_n(z)$$

Lemma 4.8. *Suppose that* m_{n+1} *is monotonous on an interval* $I_z = (z, f^q(z))$, $z \in \mathbb{T}^1$. *Then, for any* $x \in \mathbb{T}^1$, *for any* $y \in I_x$ ($I_x = (x, f^q(x))$), *we have:*

$$\left|\frac{m_{n+1}(y)}{m_{n+1}(x)} - 1\right| \le C_{29}^{f,k} M_n^{1/2}$$

with

$$C_{29}^{f,k} = 2^9(k+2)e^{(11+k/2)W(f)}(C_{17}^f)^2 C_{26}^f$$

Lemma 4.9. *If m_{n+1} is not monotonous on any interval of the form $I_z = (z, f^q(z))$, $z \in \mathbb{T}^1$, then for any $x \in \mathbb{T}^1$, $y \in I_x$, we have:*

$$|m_{n+1}(y) - m_{n+1}(x)| \leq C_{30}^{f,k} M_n^{(k-1)/2} m_n(x)$$

with

$$C_{30}^{f,k} = (C_{27}^{f}(k-1))e^{W(f)} \left(e^{(k/2+2)W(f)}(1 + e^{W(f)})^2 \frac{e^{(k/2+2)W(f)} - 1}{e^{W(f)} - 1} \right)^{k-1}$$

Proof of proposition 4.5. Let us recall the proof of proposition 4.5 from these three lemmas. (see [14, p.344]). Let $x \in \mathbb{T}^1$ and $y \in I_x$, $z \in [f^{q_{n+1}}(x), x]$ the points given by lemma 4.7. By combining lemmas 4.8 and 4.9, we obtain:

$$|m_{n+1}(y) - m_{n+1}(x)| \leq \left(\max\left(C_{29}^{f,k}, C_{30}^{f,k}\right)\right)\left(M_n^{1/2} m_{n+1}(x) + M_n^{(k-1)/2} m_n(x)\right)$$

Moreover, by lemma 3.3, we have:

$$|m_n(z) - m_n(x)| \leq C_{17}^{f} M_n^{1/2} |z - x| \leq C_{17}^{f} M_n^{1/2} m_{n+1}(x)$$

By applying lemma 4.7, and since $\alpha_{n+1}/\alpha_n \leq 1$, we get:

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \leq \left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(z) \right| + \frac{\alpha_{n+1}}{\alpha_n} |m_n(z) - m_n(x)|$$

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \leq |m_{n+1}(y) - m_{n+1}(x)| + |m_n(z) - m_n(x)|$$

Therefore, we have:

$$\left| m_{n+1}(x) - \frac{\alpha_{n+1}}{\alpha_n} m_n(x) \right| \leq C_{31}^{f,k} \left(M_n^{1/2} m_{n+1}(x) + M_n^{(k-1)/2} m_n(x) \right)$$

with $C_{31}^{f,k} = \max(C_{29}^{f,k}, C_{30}^{f,k}) + C_{17}^{f}$.
Finally, let us estimate $C_{31}^{f,k}$. Since $k \geq 3$, then:
$[4(k/2+1)(200k)]^2 \leq (k+3)^{(k+3)(k+2)k}/2$ and therefore,

$$2^{2(k-1)}(k/2+1)^{k-1}(200k)^{2(k+1)(k-1)!} \leq (k+3)^{(k+3)!}/2$$

Therefore, we have:

$$C_{30}^{f,k} + C_{17}^{f} \leq (k+3)^{(k+3)!} e^{(k+2)!W(f)} (\max(1, |Sf|_{k-3}))^{k!}$$

Since $k \geq 3$, we also have:

$$C_{29}^{f,k} + C_{17}^{f} \leq (k+3)^{(k+3)!} e^{(k+2)!W(f)} (\max(1, |Sf|_{k-3}))^{k!}$$

Therefore, $C_{31}^{f,k} \leq C_{28}^{f,k} = (k+3)^{(k+3)!} e^{(k+2)!W(f)} (\max(1, |Sf|_{k-3}))^{k!}$. Hence proposition 4.5.

□

4.2 Estimation of the C^1-conjugacy. Proof of estimation (1).

We choose an integer n_1 such that for any $n \geq n_1$, we have:

$$C_{28}^{f,k} M_n^{1/2} \leq C_{28}^{f,k} (C_{19}^f)^{\frac{n-1}{2}} < 1/2 \tag{18}$$

We take:

$$n_1 = \left\lceil \frac{-\log\left(2C_{28}^{f,k}/(C_{19}^f)^{1/2}\right)}{\log\left((C_{19}^f)^{1/2}\right)} \right\rceil$$

We choose a parameter θ such that $(k+1)/2 - \theta > (1+\beta+\theta)(1+\theta)$ (for the interpretation of this parameter θ, see the remark after proposition 4.10). We take:

$$\theta = \min\left(1/2, \left(\frac{3+\beta}{4}\right)\left(-1 + \left(1 + \frac{2(k-2\beta-1)}{(3+\beta)^2}\right)^{1/2}\right)\right) \tag{19}$$

(in the proof of estimation (2), we take $\theta = 1/2$ instead).

We recall that for $x \geq 0$, $1 + x \leq e^x$ and for $0 \leq x \leq 1/2$, $\log(1/(1-x)) \leq x/(1-x) \leq 2x$. We apply estimation (18), we use the definition of n_1 and the fact that $\theta \leq 1/2$. We get:

$$\prod_{n=n_1}^{+\infty} \left(1 + M_n^\theta\right) \leq \exp\left(\sum_{n=n_1}^{+\infty} M_n^\theta\right) \leq \exp\left(\frac{1}{2C_{28}^{f,k}(1-(C_{19}^f)^\theta)}\right)$$

$$\prod_{n=n_1}^{+\infty} \left(\frac{1}{1 - C_{28}^{f,k} M_n^{1/2}}\right) \leq \exp\left(\sum_{n=n_1}^{+\infty} 2C_{28}^{f,k} M_n^{1/2}\right) \leq \exp\left(\frac{1}{1-(C_{19}^f)^{1/2}}\right)$$

Therefore,

$$\prod_{n=n_1}^{+\infty} \left(\frac{1+M_n^\theta}{1 - C_{28}^{f,k} M_n^{1/2}}\right) \leq \exp\left(\frac{2}{1-(C_{19}^f)^\theta}\right) = C_{32} \tag{20}$$

Let:

$$C_{33} = \max\left((4C_{28}^{f,k})^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}, C_{32}\right) \tag{21}$$

For any

$$n \geq \frac{-\log\left(2(C_{33})^2\right)}{\log C_{19}^f} + 1 = C_{34} \tag{22}$$

we have:

$$M_n \leq (C_{19}^f)^{n-1} \leq \frac{1}{2C_{33}^2} \tag{23}$$

We use this estimation in the second step of the proof, to which we come now:

Let

$$n_2 = \max(n_1, \tilde{n}_2) \tag{24}$$

where $\tilde{n}_2$ is the integer defined by

$$C_{34} + \frac{4}{\log 2}\log(1/C_d) + 1 \leq \tilde{n}_2 < C_{34} + \frac{4}{\log 2}\log(1/C_d) + 2 \tag{25}$$

Having defined the integer n_2, we can present the alternative between the "favorable" case (R_n) and the "unfavorable" case (R'_n).

Proposition 4.10. *Let $a_{n_2} = 1/((C_{33})^2)$. Let $1 \geq \eta_n \geq 0$ be a sequence such that $\alpha_n = \alpha_{n+1}^{1-\eta_n}$. For any $n \geq n_2$, we can define a sequence a_n, $1/((C_{33})^2) \leq a_n \leq 1/C_{33}$ and a sequence $\rho_n < 1$ such that $M_n = a_n\alpha_n^{\rho_n}$. The sequence a_n is defined by:*

if

$$(R_n) \quad C_{28}^{f,k} M_n^{(k+1)/2-\theta} \leq M_n \frac{\alpha_{n+1}}{\alpha_n} \quad \text{then} \quad a_{n+1} = a_n \frac{1 + M_n^\theta}{1 - C_{28}^{f,k} M_n^{1/2}}$$

and if

$$(R'_n) \quad C_{28}^{f,k} M_n^{(k+1)/2-\theta} > M_n \frac{\alpha_{n+1}}{\alpha_n} \quad \text{then} \quad a_{n+1} = a_n$$

Moreover, if (R_n) holds, then $\rho_{n+1} \geq \rho_n + \eta_n(1 - \rho_n)$;

and if (R'_n) holds, then $\rho_{n+1} \geq ((k+1)/2 - \theta)(1 - \eta_n)\rho_n$. In particular, the sequence $(\rho_n)_{n \geq n_2}$ is increasing.

The threshold between the alternatives (R_n) and (R'_n) is controlled with a parameter θ, which could be freely chosen such that $\theta > 0$ and $(k+1)/2 - \theta \geq (1 + \beta + \theta)(1 + \theta)$. When θ increases, the number n_3 of occurrences of (R'_n) increases. When n_3 increases, all other quantities being equal, the bound on the norm of the conjugacy increases. Moreover, if θ gets too large, we can no longer show that n_3 is finite (see proposition 4.14), and therefore, we can no longer estimate the norm of the conjugacy.

On the other hand, when θ is smaller, C_{32} increases. It increases the number n_2 above which we consider the alternatives (R_n) and (R'_n). C_{35} increases too (see proposition 4.19). When C_{32} and C_{35} increase, all other quantities being equal, the bound on the norm of the conjugacy increases. Moreover, when $\theta \to 0$, $C_{32} \to +\infty$, which makes this bound on the conjugacy diverge.

Thus, the variation of θ has contradictory influences on the bound of the norm of the conjugacy, and there is a choice of θ that optimizes this bound. However, in this paper, we do not seek this optimal θ, since it would complicate further the expression of the final estimate. Instead, in estimation (2), we fix $\theta = 1/2$, which allows simplifying the expression of the estimate. In estimation (3), we take $\theta \to 0$, which also allows simplifying the estimate.

Proof of proposition 4.10: For any $n \geq n_2$, since $n_2 \geq n_1$,

$$a_{n_2} = \frac{1}{C_{33}^2} \leq a_n \leq a_{n_2} \prod_{n=n_1}^{+\infty} \left(\frac{1 + M_n^\theta}{1 - C_{28}^{f,k} M_n^{1/2}} \right) \leq \frac{C_{32}}{C_{33}^2} \leq \frac{1}{C_{33}}$$

and since

$$\alpha_n^{\rho_n} > a_n\alpha_n^{\rho_n} = M_n \geq \alpha_n$$

then $\rho_n < 1$.
Second, if (R_n) holds, then by applying corollary 4.6, we have:

$$M_{n+1} \leq \frac{1 + M_n^\theta}{1 - C_{28}^{f,k} M_n^{1/2}} M_n \frac{\alpha_{n+1}}{\alpha_n} \tag{26}$$

Therefore,

$$M_{n+1} = a_{n+1}\alpha_{n+1}^{\rho_{n+1}} \leq a_{n+1}\alpha_{n+1}\alpha_n^{\rho_n - 1} = a_{n+1}\alpha_{n+1}\alpha_{n+1}^{(1-\eta_n)(\rho_n-1)}$$

and then:

$$\rho_{n+1} - 1 \geq (1 - \eta_n)(\rho_n - 1)$$

hence the estimation:

$$\rho_{n+1} \geq \rho_n + \eta_n(1 - \rho_n)$$

If (R_n') holds, since $C_{28}^{f,k} M_n^{1/2} \leq 1/2$, then by applying corollary 4.6, we obtain:

$$M_{n+1} \leq 4C_{28}^{f,k} M_n^{(k+1)/2-\theta}$$

Moreover, since $a_n \leq 1/C_{33} < 1$, then:

$$a_n^{(k+1)/2-\theta} \leq a_n^{(1+\beta+\theta)(1+\theta)} = a_n a_n^{(1+\beta+\theta)(1+\theta)-1} \leq \frac{a_n}{C_{33}^{(1+\beta+\theta)(1+\theta)-1}} \leq \frac{a_n}{4C_{28}^{f,k}}$$

Therefore, by combining these two estimations, we obtain:

$$a_{n+1}\alpha_{n+1}^{\rho_{n+1}} = M_{n+1} \leq 4C_{28}^{f,k} M_n^{(k+1)/2-\theta} \leq 4C_{28}^{f,k} a_n^{(k+1)/2-\theta} \alpha_n^{\rho_n((k+1)/2-\theta)} \leq a_n \alpha_n^{\rho_n((k+1)/2-\theta)}$$

Moreover, since $a_{n+1} = a_n$, then

$$1 \leq \alpha_{n+1}^{(\rho_n((k+1)/2-\theta))(1-\eta_n)-\rho_{n+1}}$$

hence the estimation:

$$\rho_{n+1} \geq (\rho_n((k + 1)/2 - \theta))(1 - \eta_n)$$

□

The reader can notice that until now, we have not used the Diophantine condition on α yet. Now, we introduce this condition in order to estimate ρ_{n_2} from below (proposition 4.11), and in order to determine a bound ρ above which (R_n) always occurs (proposition 4.12).

Proposition 4.11. *If $\beta > 0$, we have the estimation:*

$$\rho_{n_2} \geq \frac{\log 2}{((1+\beta)^{n_2+1}-1)\log(1/C_d)/\beta}$$

If $\beta = 0$, we have the estimation:

$$\rho_{n_2} \geq \frac{\log 2}{(n_2+1)\log(1/C_d)}$$

Proof. Since α is Diophantine, we have: $\alpha_{n+1} \geq C_d \alpha_n^{1+\beta}$. Therefore, for $\beta > 0$,

$$\log\left(\frac{1}{\alpha_{n+1}}\right) + \frac{\log(1/C_d)}{\beta} \leq (1+\beta)\left(\log(1/\alpha_n) + \frac{\log(1/C_d)}{\beta}\right)$$

and since $\alpha_{-1} = 1$, then by iteration, for any $n \geq 0$,

$$\log(1/\alpha_n) \leq \left((1+\beta)^{n+1} - 1\right)\frac{\log(1/C_d)}{\beta}$$

If $\beta = 0$, we have:

$$\log(1/\alpha_n) \leq (n+1)\log(1/C_d)$$

Moreover, since $\rho_{n_2} = -\log(M_{n_2}/a_{n_2})/\log(1/\alpha_{n_2})$ and $M_{n_2}/a_{n_2} \leq 1/2$, then we get proposition 4.11.

□

Proposition 4.12. *Let $\beta_1 = \beta + \frac{2\log(1/C_d)}{(n_2-1)\log 2}$. If*

$$\rho_n \geq \frac{\beta_1}{(k-1)/2-\theta} = \rho \tag{27}$$

then (R_n) occurs.

Remark 4.13. Note that $\rho < 1$, because $(k+1)/2-\theta \geq (1+\beta+\theta)(1+\theta)$ and $\beta_1 \leq \beta + 1/2$.

Proof. Since $\alpha_n \leq (1/2)^{\frac{n-1}{2}}$, then

$$0 < \frac{\log C_d}{\log \alpha_n} \leq \frac{-\log C_d}{\frac{n-1}{2}\log 2} \tag{28}$$

Furthermore, since $\alpha_{n+1} = \alpha_n^{\frac{1}{1-\eta_n}} \geq C_d \alpha_n^{1+\beta}$, then

$$\frac{1}{1-\eta_n}\log\alpha_n \geq \log C_d + (1+\beta)\log\alpha_n$$

and since $\log\alpha_n < 1$ for $n \geq 0$, then by (28),

$$\frac{1}{1-\eta_n} - 1 \leq \beta + \frac{\log C_d}{\log\alpha_n} \leq \beta + \frac{\log(1/C_d)}{\frac{n-1}{2}\log 2}$$

Therefore, if estimation (27) holds, then

$$\left(\frac{k-1}{2} - \theta\right)\rho_n + 1 - \frac{1}{1-\eta_n} \geq 0$$

and therefore,

$$\left(\frac{1}{\alpha_n}\right)^{(\frac{k-1}{2}-\theta)\rho_n+1-\frac{1}{1-\eta_n}} \geq 1$$

Hence

$$M_n \frac{\alpha_{n+1}}{\alpha_n} = a_n \alpha_n^{\rho_n} \frac{\alpha_{n+1}}{\alpha_n} \geq a_n \alpha_n^{(\frac{k+1}{2}-\theta)\rho_n} = M_n^{\frac{k+1}{2}-\theta} a_n^{1-(\frac{k+1}{2}-\theta)} \geq M_n^{\frac{k+1}{2}-\theta} C_{33}^{\frac{k+1}{2}-\theta-1} \geq M_n^{\frac{k+1}{2}-\theta} C_{33}^{(1+\beta+\theta)(1+\theta)-1}$$

Therefore,

$$M_n \frac{\alpha_{n+1}}{\alpha_n} \geq C_{28}^{f,k} M_n^{\frac{k+1}{2}-\theta}$$

$\square$

Proposition 4.14. *The alternative (R'_n) occurs less than n_3 times, with*

$$n_3 - n_2 \leq \max\left(0, \frac{\log(\rho/\rho_{n_2})}{\log\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)}\right) \tag{29}$$

Proof. If $\rho_{n_2} \geq \rho$, then (R'_n) does not occur for any $n \geq n_2$. We suppose $\rho_{n_2} < \rho$. For any $n \geq n_2$, since

$$((k+1)/2 - \theta)(1-\eta_n) \geq \frac{(k+1)/2 - \theta}{1+\beta_1}$$

then

$$\rho_n \geq \left(\frac{(k+1)/2 - \theta}{1+\beta_1}\right)^{n-n_2} \rho_{n_2}$$

Moreover,

$$\left(\frac{(k+1)/2 - \theta}{1+\beta_1}\right)^{n-n_2} \rho_{n_2} \geq \rho$$

when

$$n \geq n_2 + \frac{\log(\rho/\rho_{n_2})}{\log\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)}$$

$\square$

The next proposition gives a lower bound on α_{n_4}, which allows computing a bound on the C^1-conjugacy.

Proposition 4.15. *Let $n_4 \geq 0$ be the smallest integer such that for any $n \geq n_4$, (R_n) occurs. We have:*

$$\alpha_{n_4} \geq C_d^{\exp((n_3+1+\rho/(1-\rho))(1+\beta_1))}$$

Proof. First, we suppose $n_4 \geq n_2 + 1$ We need the lemma:

Lemma 4.16. *Let $n_5 \geq n_2$ be the smallest integer such that*

$$\sum_{n=n_2}^{n_5} \eta_n \geq n_3 - n_2 + \rho/(1-\rho)$$

n_5 exists. Moreover, we have $\rho_{n_5+1} \geq \rho$. In particular, for this integer n_5, we have that for any $n \geq n_5 + 1$, (R_n) occurs.

Proof. First, let us show the existence of n_5. By absurd, suppose that

$$\sum_{n=n_2}^{+\infty} \eta_n < n_3 - n_2 + \rho/(1-\rho)$$

For any $1 > x \geq 0$,

$$\log\left(\frac{1}{1-x}\right) \leq \frac{x}{1-x}$$

Therefore, for any integer $p \geq n_2 + 1$,

$$\prod_{n=n_2}^{p-1}\left(\frac{1}{1-\eta_n}\right) \leq \exp\left(\sum_{n=n_2}^{p-1} \frac{\eta_n}{1-\eta_n}\right)$$

Moreover, $\frac{1}{1-\eta_n} \leq 1 + \beta_1$ for any $n \geq 1$. Therefore,

$$\sum_{n=n_2}^{p-1} \frac{\eta_n}{1-\eta_n} \leq (n_3 - n_2 + \rho/(1-\rho))(1+\beta_1)$$

Since $\eta_n \leq 1$, then $\sum_{n=0}^{n_2-1} \eta_n \leq n_2$. Therefore,

$$\sum_{n=0}^{p-1} \frac{\eta_n}{1-\eta_n} \leq (n_3 + \rho/(1-\rho))(1+\beta_1)$$

Moreover, since $\alpha_0 = \alpha \geq C_d$ then for any $p \geq n_2 + 1$:

$$\alpha_p = \alpha_0^{\prod_{n=0}^{p-1}\left(\frac{1}{1-\eta_n}\right)} \geq C_d^{\exp((n_3+\rho/(1-\rho))(1+\beta_1))}$$

However, since $\alpha_p \geq 2\alpha_{p+2}$, then $\alpha_p \to 0$ when $p \to +\infty$. Hence the contradiction and the existence of n_5. Note that $n_5 + 1 \geq n_4$.

Second, let us show that $\rho_{n_5+1} \geq \rho$. If there is $n_6 \leq n_5$ such that $\rho_{n_6} \geq \rho$, then $\rho_{n_5+1} \geq \rho$ because the sequence ρ_n is increasing. Otherwise, for any $n \leq n_5$, we have: $\rho_n \leq \rho$.

Let $E_1 = \{n_5 \geq n \geq n_2 \ / \ (R_n) \text{ occurs}\}$ and $E_2 = \{n_5 \geq n \geq n_2 \ / \ (R'_n) \text{ occurs}\}$.

We have:

$$n_3 - n_2 + \frac{\rho}{1-\rho} \leq \sum_{n=n_2}^{n_5} \eta_n = \sum_{n \in E_1} \eta_n + \sum_{n \in E_2} \eta_n \leq \sum_{n \in E_1} \eta_n + n_3 - n_2$$

Therefore,

$$\sum_{n\in E_1} \eta_n \geq \rho/(1-\rho)$$

Since ρ_n is increasing and $\rho_n \leq \rho$, we get:

$$\rho_{n_5+1} = \rho_{n_2} + \sum_{n=n_2}^{n_5} \rho_{n+1} - \rho_n$$

$$\rho_{n_5+1} \geq \rho_{n_2} + \sum_{n\in E_1} \rho_{n+1} - \rho_n \geq \rho_{n_2} + \sum_{n\in E_1}(1-\rho_n)\eta_n \geq \rho_{n_2} + (1-\rho)\sum_{n\in E_1}\eta_n \geq \rho$$

□

Now, let us show proposition 4.15. Since $\eta_n \leq 1$ for any n, then we have:

$$n_3 - n_2 + 1 + \frac{\rho}{1-\rho} > \sum_{n=n_2}^{n_5} \eta_n \geq n_3 - n_2 + \frac{\rho}{1-\rho}$$

Since

$$n_3 - n_2 + \frac{\rho}{1-\rho} + 1 \geq \sum_{n=n_2}^{n_5} \eta_n \geq \sum_{n=n_2}^{n_4-1} \eta_n$$

then by proceeding in the same way as in the first part of the proof of lemma 4.16, we obtain:

$$\alpha_{n_4} \geq C_d^{\exp((n_3+1+\rho/(1-\rho))(1+\beta_1))} \tag{30}$$

Finally, if $n_4 \leq n_2$, then as in the proof of lemma 4.16,

$$\alpha_{n_2} = \alpha_0^{\prod_{n=0}^{n_2-1}\left(\frac{1}{1-\eta_n}\right)} \geq C_d^{\exp(n_2(1+\beta_1))}$$

Therefore, the estimation given in proposition 4.15 still holds.

□

Having bounded α_{n_4} from below, we show how this bound is related to M_n/m_n (and therefore, how this is related to the conjugacy).

Proposition 4.17. *Let $n \geq 1$, $M_n = \max_{x\in\mathbb{T}^1} m_n(x)$, $m_n = \min_{x\in\mathbb{T}^1} m_n(x)$. For any $j \leq n$,*

$$\frac{M_j}{m_j} \leq 3|Df|_0^{\frac{2}{\alpha_n}}$$

Proof. We need the following lemma, which is in [14, p. 339]:

Lemma 4.18. *For any $x \in \mathbb{T}^1$, let $J_x = [f^{-q_n}(x), f^{q_n}(x)]$. The intervals $f^i(J_x), 0 \leq i < q_{n+1}$ recover $\mathbb{T}^1$.*

First, note that since $f(x+1) - f(x) = 1$ (in the universal cover), then $|Df|_0 \geq 1$. Let $x, y \in \mathbb{T}^1$ such that $M_n = m_n(x)$ and $m_n = m_n(y)$. Let $0 \leq i < q_{n+1}$ such that $x \in f^i(J_y)$. Since we have the cyclic order $f^{i-q_n}(y) \leq x \leq f^{i+q_n}(y)$ then we also have: $f^i(y) \leq f^{q_n}(x) \leq f^{i+2q_n}(y)$. Therefore, $[x, f^{q_n}(x)] \subset [f^{i-q_n}(y), f^{i+2q_n}(y)]$. This implies:

$$M_n \leq f^{i+2q_n}(y) - f^{i+q_n}(y) + f^{i+q_n}(y) - f^i(y) + f^i(y) - f^{i-q_n}(y)$$

$$M_n \leq \left(|Df^{i+q_n}|_0 + |Df^i|_0 + |Df^{i-q_n}|_0\right)(f^{q_n}(y) - y)$$

and therefore,

$$\frac{M_n}{m_n} \leq \left(|Df^{i+q_n}|_0 + |Df^i|_0 + |Df^{i-q_n}|_0\right)$$

Therefore,

$$\frac{M_n}{m_n} \leq 3|Df|_0^{q_n+q_{n+1}}$$

Likewise, for any $j \leq n$, we have:

$$\frac{M_j}{m_j} \leq 3|Df|_0^{q_n+q_{n+1}}$$

Since $q_n + q_{n+1} \leq 2q_{n+1} \leq \frac{2}{\alpha_n}$, we obtain proposition 4.17.

□

Proposition 4.19. *For any $n \geq 1$,*

$$\frac{M_n}{m_n} \leq C_{35} \frac{M_{n_4}}{m_{n_4}} \tag{31}$$

with:

$$C_{35} = \exp\left(\frac{2(2C_{33}^2)^\theta - 1}{(2C_{33}^2)^\theta - 1} \frac{(C_{19}^f)^{(n_2-1)\theta}}{1 - (C_{19}^f)^\theta} + 3C_{28}^{f,k} \frac{(C_{19}^f)^{(n_2-1)/2}}{1 - (C_{19}^f)^{1/2}}\right) \tag{32}$$

Proof. Since for any $n \geq n_4$, (R_n) occurs, then by corollary 4.6, we have:

$$\frac{M_{n+1}}{M_n} \leq \frac{1 + M_n^\theta}{1 - C_{28}^{f,k} M_n^{1/2}} \frac{\alpha_{n+1}}{\alpha_n}$$

$$\frac{m_{n+1}}{m_n} \geq \frac{1 - M_n^\theta}{1 + C_{28}^{f,k} M_n^{1/2}} \frac{\alpha_{n+1}}{\alpha_n}$$

Therefore,

$$\frac{M_{n+1}/m_{n+1}}{M_n/m_n} \leq \frac{1 + M_n^\theta}{1 - M_n^\theta} \frac{1 + C_{28}^{f,k} M_n^{1/2}}{1 - C_{28}^{f,k} M_n^{1/2}} \tag{33}$$

Therefore, for any $n \geq n_4$,

$$\frac{M_n}{m_n} \le \frac{M_{n_4}}{m_{n_4}} \prod_{j=n_4}^{+\infty} \frac{1+M_j^\theta}{1-M_j^\theta} \frac{1+C_{28}^{f,k} M_j^{1/2}}{1-C_{28}^{f,k} M_j^{1/2}}$$

As in the proof of estimation (20), and since $n_4 \ge n_2$, we have:

$$\prod_{j=n_4}^{+\infty} \left(1+M_j^\theta\right) \le \exp\left(\sum_{j=n_2}^{+\infty} M_j^\theta\right) \le \exp\left(\frac{(C_{19}^f)^{(n_2-1)\theta}}{1-(C_{19}^f)^\theta}\right)$$

$$\prod_{j=n_4}^{+\infty} \left(1+C_{28}^{f,k} M_j^{1/2}\right) \le \exp\left(C_{28}^{f,k} \sum_{j=n_2}^{+\infty} M_j^{1/2}\right) \le \exp\left(C_{28}^{f,k} \frac{C_{19}^{f(n_2-1)/2}}{1-(C_{19}^f)^{1/2}}\right)$$

$$\prod_{j=n_4}^{+\infty} \frac{1}{1-C_{28}^{f,k} M_j^{1/2}} \le \exp\left(C_{28}^{f,k} \sum_{j=n_2}^{+\infty} \frac{M_j^{1/2}}{1-C_{28}^{f,k} M_j^{1/2}}\right) \le \exp\left(2C_{28}^{f,k} \frac{(C_{19}^f)^{(n_2-1)/2}}{1-(C_{19}^f)^{1/2}}\right) \quad (34)$$

and since, for $j \ge n_2$, $M_j \le 1/(2C_{33}^2)$, we get:

$$\prod_{j=n_4}^{+\infty} \frac{1}{1-M_j^\theta} \le \exp\left(\sum_{j=n_2}^{+\infty} \frac{M_j^\theta}{1-M_j^\theta}\right) \le \exp\left(\frac{\frac{(C_{19}^f)^{(n_2-1)\theta}}{1-(C_{19}^f)^\theta}}{1-\frac{1}{(2C_{33}^2)^\theta}}\right)$$

Therefore,

$$\frac{M_n}{m_n} \le \frac{M_{n_4}}{m_{n_4}} \exp\left(\frac{2(2C_{33}^2)^\theta - 1}{(2C_{33}^2)^\theta - 1} \frac{(C_{19}^f)^{(n_2-1)\theta}}{1-(C_{19}^f)^\theta} + 3C_{28}^{f,k} \frac{(C_{19}^f)^{(n_2-1)/2}}{1-(C_{19}^f)^{1/2}}\right) \quad (35)$$

Hence proposition 4.19.

□

Proof of estimation (1). By combining propositions 4.17 and 4.19, and since by [14, p. 348],
$|Dh|_0 \le \sup_{n\ge 0} M_n/m_n$, we get:

$$|Dh|_0 \le C_{36} |Df|_0^{\frac{2}{\alpha_{n_4}}} \quad (36)$$

with:

$$C_{36} = 3C_{35}$$

We estimate C_{35}: since $(2x-1)/(x-1) = 2 + 1/(x-1)$, since $(C_{19}^f)^{(n_2-1)\theta} \le 1/(2(C_{33})^2)^\theta$ and since $\theta \le 1/2$, then:

$$C_{35} \le \exp\left(\left(2 + \frac{1}{(2(C_{33})^2)^\theta - 1} + 3C_{28}^{f,k}\right) \frac{1}{(2(C_{33})^2)^\theta (1-(C_{19}^f)^\theta)}\right)$$

Since $C_{19}^f \ge 1$, we get:

$$|Dh|_0 \leq C_{37}|Df|_0^{\frac{2}{\alpha_{n_4}}} \tag{37}$$

with:

$$C_{37} = 3e \wedge \left(\left(2 + \frac{1}{(2(C_{33})^2)^\theta - 1} + 3C_{28}^{f,k} \right) \frac{1}{(2(C_{33})^2)^\theta (1 - (C_{19}^f)^\theta)} \right)$$

We estimate C_{37} using expressions of θ (see (19)), of C_{19}^f (see lemma 3.4) and of C_{33} (see (21) and proposition 4.5).

We estimate α_{n_4} using propositions 4.15, 4.14, 4.12, 4.11, and the expressions of n_2 (see (24)) and estimates of θ, C_{19}^f and C_{33}. We get:

$$|Dh|_0 \leq C_2(k, \beta, C_d, |Df|_0, W(f), |Sf|_{k-3})$$

where C_2 is the combination of the following functions:

1. $C_{19}^f = \left(1 + e \wedge \left(-6\sqrt{2}e^{2W(f)}(|Sf|_0^{1/2} \vee 1)\right)\right)^{-1/2}$
   (since $|Sf|_0 \leq |Sf|_{k-3}$, we can estimate C_{19}^f in function of $W(f), |Sf|_{k-3}$).
2. $C_{28}^{f,k} = (k+3)^{(k+3)!} e^{(k+3)!W(f)} (\max(1, |Sf|_{k-3}))^{(k+1)!}$
3. $\theta = \min\left(1/2, \left(\frac{3+\beta}{4}\right)\left(-1 + \sqrt{1 + \frac{2(k-(2\beta+1))}{(3+\beta)^2}}\right)\right)$
4. $C_{33} = \max\left(e^{\frac{2}{1-(C_{19}^f)^\theta}}, (4C_{28}^{f,k})^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}\right)$
5. $n_2 = \lfloor \max\left(-\frac{\log(2C_{33}^2)}{\log C_{19}^f} + \frac{2\log(1/C_d)}{\theta \log 2} + 2, 2 + \frac{(2C_{28}^{f,k})}{\log((C_{19}^f)^{1/2})}\right) \rfloor$
6. $\beta_1 = \beta + \frac{2\log(1/C_d)}{(n_2-1)\log 2}$
7. $n_3 = \lceil \frac{1}{\log\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right)} \left(n_2(1 + \log(1+\beta)) + \log\left(\frac{(n_2+1)\log(1/C_d)}{\log 2}\right)\right) \rceil$
8. $\rho = \frac{\beta_1}{\frac{k-1}{2} - \theta}$
9. $\alpha'_{n_4} = C_d \wedge \left(e \wedge \left(\left(n_3 + 1 + \frac{\rho}{1-\rho}\right)(1+\beta_1)\right)\right)$
10. $C_{37} = 3e \wedge \left(\left(2 + \frac{1}{(2(C_{33})^2)^\theta - 1} + 3C_{28}^{f,k}\right) \frac{1}{(2(C_{33})^2)^\theta(1-(C_{19}^f)^\theta)}\right)$
11. $|Dh|_0 \leq C_{37}|Df|_0^{\frac{2}{\alpha'_{n_4}}}$

Note that we have a bound $\alpha'_{n_4} \leq \alpha_{n_4}$, but we do not know the value of α_{n_4}.

$\square$

4.3 Proof of estimation (2).

In order to obtain relatively simple estimates, we can take the parameter θ (defined in (19)) either vanishingly close to 0 (estimation (3)), or fixed independently of the other parameters (estimation (2)).

In the latter case, we need to assume that $k - 2\beta - 1$ is sufficiently large, in order to keep $(k+1)/2 - \theta \geq (1 + \beta + \theta)(1 + \theta)$. To illustrate this case, we take $\theta = 1/2$, which requires $k \geq 3\beta + 9/2$ (for any fixed θ, we cannot obtain an assumption of the form $k \geq 2\beta + u$ for some number u: we necessarily have $k \geq \lambda\beta + u$ with $\lambda > 2$).

To simplify the function C_2, we successively estimate C_{35}, α'_{n_4} and n_2.

Lemma 4.20. *Let C_{35} and C_{33} defined in proposition 4.19 and (21) respectively. We have:*

$$C_{35} \leq \exp\left(C_{33}^{\frac{3\beta+1}{2}}\right)$$

Proof. For any $x \geq 2$,

$$(2x-1)/(x-1) = 2 + 1/(x-1) \leq 3$$

Since $C_{33} \geq C_{32} \geq e \geq 2$, then

$$\exp\left(\tfrac{2(2C_{33})^{1/2}-1}{(2C_{33})^{1/2}-1}(1/C_{19}^f)^{1/2}\tfrac{(C_{19}^f)^{n_2/2}}{1-(C_{19}^f)^{1/2}} + 3C_{28}^{f,k}(1/C_{19}^f)^{1/2}\tfrac{(C_{19}^f)^{n_2/2}}{1-(C_{19}^f)^{1/2}}\right)$$
$$\leq \exp\left(3(1 + C_{28}^{f,k})(1/C_{19}^f)^{1/2}\tfrac{(C_{19}^f)^{n_2/2}}{1-(C_{19}^f)^{1/2}}\right)$$

On the other hand, since $n_2 \geq \frac{-\log\left(2(C_{33})^2/C_{19}^f\right)}{\log C_{19}^f}$ and $C_{19}^f < 1$, then:

$$e^{\frac{n_2}{2}\log C_{19}^f} \leq \left(2(C_{33})^2/C_{19}^f\right)^{-1/2}$$

Therefore,

$$(1/C_{19}^f)^{1/2}\frac{(C_{19}^f)^{n_2/2}}{1-(C_{19}^f)^{1/2}} = (1/C_{19}^f)^{1/2}\frac{e^{\frac{n_2}{2}\log C_{19}^f}}{1-(C_{19}^f)^{1/2}}$$

$$\leq (1/C_{19}^f)^{1/2}\frac{\left(2(C_{33})^2/C_{19}^f\right)^{-1/2}}{1-(C_{19}^f)^{1/2}} = \frac{1}{\sqrt{2}C_{33}(1-(C_{19}^f)^{1/2})}$$

Moreover, since $C_{28}^{f,k} \geq 1/(\sqrt{2}-1)$ then $(1 + C_{28}^{f,k})/\sqrt{2} \leq C_{28}^{f,k}$. Therefore,

$$\exp\left(3(1 + C_{28}^{f,k})(1/C_{19}^f)^{1/2}\frac{(C_{19}^f)^{n_2/2}}{1-(C_{19}^f)^{1/2}}\right) \leq \exp\left(\frac{3C_{28}^{f,k}}{C_{33}(1-(C_{19}^f)^{1/2})}\right)$$

Since $C_{33}^{\frac{3}{2}(3/2+\beta)-1} \geq 4C_{28}^{f,k}$, then

$$\frac{C_{28}^{f,k}}{C_{33}} \leq \frac{1}{4}C_{33}^{\frac{1}{4}+\frac{3}{2}\beta}$$

We also have:

$$C_{33}^{\frac{1}{4}} \geq e^{\frac{1/2}{1-(C_{19}^f)^{1/2}}} \geq \frac{1}{4}\frac{1}{1-(C_{19}^f)^{1/2}}$$

We obtain:

$$\exp\left(\frac{3C_{28}^{f,k}}{C_{33}(1-(C_{19}^f)^{1/2})}\right) \leq \left(e^{\frac{1}{4}C_{33}^{\frac{1}{4}+\frac{3}{2}\beta}\frac{1}{1-(C_{19}^f)^{1/2}}}\right) \leq \exp\left(C_{33}^{\frac{3\beta+1}{2}}\right)$$

□

Lemma 4.21. *Let α'_{n_4} defined page 23 and n_2 defined in (24). We have:*

$$\frac{1}{\alpha'_{n_4}} \leq \left(\frac{1}{C_d}\right) \wedge e \wedge \left((\beta+3/2)\left(2+\frac{n_2}{\log(3/2)}\left(2+\log(1+\beta)+\log\log(1/C_d)\right)\right)\right)$$

Proof. Since $\beta_1 \leq \beta + 1/2$, and $\theta = 1/2$, then

$$\log\left(\frac{(k+1)/2-\theta}{1+\beta_1}\right) \geq \log(3/2)$$

Therefore,

$$n_3 \leq \frac{n_2}{\log(3/2)}\left(1+\log(1+\beta)+\frac{\log\left((n_2+1)\log(1/C_d)\right)}{n_2\log 2}\right)$$

We have $\log(n_2+1)/n_2 \leq 1$ and $n_2\log 2 \geq 1$.
Moreover, $\rho \leq 2/3$, and so $\rho/(1-\rho) \leq 2$. Hence the lemma.

□

Lemma 4.22. *Let n_2 defined in (24). We have:*

$$n_2 \leq C_{38}(W(f), |Sf|_0)(k+4)!(1+\log(\max(1,|Sf|_{k-3})))(1+\log(1/C_d))$$

with:

$$C_{38}(W(f), |Sf|_0) = e^{(2)} \wedge (3W(f) + 2\log(\max(1,|Sf|_0)) + 4)$$

Proof. In order to estimate n_2, we need to estimate C_{33} (see page 23). We distinguish the cases $C_{33} = (4C_{28}^{f,k})^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}$ and $C_{33} = C_{32}^{f,k}$.

First, we suppose $C_{33} = (4C_{28}^{f,k})^{\frac{1}{(1+\beta+\theta)(1+\theta)-1}}$. We have: $C_{33} \leq 4C_{28}^{f,k}$. Therefore,

$$n_2 \leq 2 + \frac{\log\left(2(4C_{28}^{f,k})^2\right)}{-\log(C_{19}^f)} + \frac{4}{\log 2}\log(1/C_d)$$

Moreover (see page 23),

$$\log(C_{28}^{f,k}) \leq (k+3)!\left[\log(k+3) + W(f) + \max(1,|Sf|_{k-3})\right]$$

And for any $x > -1$, $\log(1+x) \geq x/(1+x)$. Therefore,

$$\frac{1}{-\log(C_{19}^f)} \leq \frac{1}{\frac{1}{2}\log(1+e^{-C_{22}^f})} \leq 2 + 2e^{C_{22}^f}$$

Therefore,

$$n_2 \leq 2 + 2\left(1 + e^{C_{22}^f}\right)[5(\log 2) + 2(k+3)!\,(W(f) + \log(k+3) + \log(\max(1, |Sf|_{k-3})))]$$

$$+\frac{4}{\log 2}\log(1/C_d)$$

Moreover, by relation (11),

$$4e^{C_{22}^f} \leq \exp\left(\exp\left(2W(f) + \log(6\sqrt{2}(\max(1, |Sf|_0))^{1/2}) + 1\right)\right) = C_{39}^f$$

Moreover, $(k+4)!/3 \geq 2 + 5\log 2$ and $2(k+4)/3 \geq 2\log(k+3)$. Therefore,

$$(k+4)! \geq 2 + 5\log 2 + 2(k+3)!\log(k+3)$$

Moreover, $2\left(1 + e^{C_{22}^f}\right) \leq 4e^{C_{22}^f}$. We get:

$$n_2 \leq (k+4)!C_{39}^f(W(f) + 1 + \log(\max(1, |Sf|_{k-3})) + \frac{4}{\log 2}\log(1/C_d)$$

By using that $\log(6\sqrt{2}) + 1 \leq 4$ and $4/\log 2 \leq (k+4)!$, we obtain the estimation of lemma 4.22.

If $C_{33} = C_{32}$, we need the following lemma:

Lemma 4.23. *Let C_{33} given by (21) and C_{34} given by (22). If $C_{33} = C_{32}$, then*

$$C_{34} \leq 40e^{2C_{22}^f}$$

Proof. We have:

$$C_{34} = \frac{-\log\left(2(C_{33})^2/C_{19}^f\right)}{\log C_{19}^f} = 1 + \frac{\log 2}{\frac{1}{2}\log(1 + e^{-C_{22}^f})} + \frac{4}{(1 - (1 + e^{-C_{22}^f})^{-1/4})\frac{1}{2}\log(1 + e^{-C_{22}^f})}$$

Since for any $x > -1$, $\log(1+x) \geq x/(1+x)$, then:

$$\frac{1}{\log(1 + e^{-C_{22}^f})} \leq e^{C_{22}^f}(1 + e^{-C_{22}^f})$$

On the other hand,

$$1 - (1 + e^{-C_{22}^f})^{-1/4} = \int_0^{e^{-C_{22}^f}} \frac{1}{4}\frac{1}{(1+t)^{5/4}}\,dt \geq \frac{e^{-C_{22}^f}}{4(1 + e^{-C_{22}^f})^{5/4}}$$

Therefore,

$$C_{34} \leq 8(1 + e^{-C_{22}^f})^{9/4}e^{2C_{22}^f}\left(\frac{1}{8(1 + e^{-C_{22}^f})e^{2C_{22}^f}} + \frac{\log 2}{4e^{C_{22}^f}} + 4\right)$$

Since $C_{22}^f \geq 6\sqrt{2}$ (see expression (11)), then:

$$e^{-C_{22}^f} \leq 2.07 \times 10^{-4} \tag{38}$$

Therefore,

$$\frac{1}{8(1+e^{-C_{22}^f})e^{2C_{22}^f}} + \frac{\log 2}{4e^{C_{22}^f}} + 4 \leq 4.1$$

Therefore,

$$C_{34} \leq 40e^{2C_{22}^f}$$

$\square$

Assuming that $C_{33} = C_{32}$, if $n_2 = \lfloor 2 + \frac{(2C_{28}^{f,k})}{\log((C_{19}^f)^{1/2})} \rfloor$ then we can follow the first part of the proof and we still obtain the estimate of lemma 4.22.

If $n_2 = \lfloor -\frac{\log(2C_{33}^2)}{\log C_{19}^f} + \frac{2\log(1/C_d)}{\theta \log 2} + 2 \rfloor$, then:

$$\frac{-\log\left(2(C_{33})^2/C_{19}^f\right)}{\log C_{19}^f} = C_{34} \leq 20e^{2C_{22}^f} \leq 2(C_{39}^f)^2 \leq C_{38}^f$$

Therefore,

$$n_2 \leq -\frac{\log(2C_{33}^2)}{\log C_{19}^f} + \frac{2\log(1/C_d)}{\theta \log 2} + 2 \leq C_{38}^f + \frac{4\log(1/C_d)}{\log 2} + 1$$

The estimation of lemma 4.22 still holds. This completes the proof of lemma 4.22.

$\square$

Now, we show estimation (2). We denote $\log^{(2)}(x) = \log\log x$. By applying estimation (36) and lemma 4.21, we have:

$$|Dh|_0 \leq e^{(3)} \wedge \left(\log^{(2)}(C_{36}) + \log^{(2)}(|Df|_0) + \log 2 + \log^{(2)}(1/C_d) + (\beta + 3/2)\left(2 + \frac{n_2}{\log(3/2)}\left(2 + \beta + \log^{(2)}(1/C_d)\right)\right)\right)$$

By lemma 4.20, we have:

$$\log^{(2)}(C_{36}) \leq \log^{(2)}(3C_{35}) \leq \log\left(\log 3 + C_{33}^{\frac{3\beta+1}{2}}\right)$$

Moreover, since $C_{33} \geq e$, $\frac{3\beta+1}{2} \geq 1/2$ and $e(e^{1/2} - 1) \geq \log 3$, then

$$\log 3 + C_{33}^{\frac{3\beta+1}{2}} \leq C_{33}^{\frac{3\beta+1}{2}+1}$$

Therefore,

$$\log^{(2)}(C_{36}) \leq \left(\frac{3\beta+1}{2} + 1\right) C_{33}$$

Moreover, by estimation (38), $e^{-C_{22}^f} \leq 2.07 \times 10^{-4}$, and therefore, $-\log(C_{19}^f) \leq 2/3$ (see expression (10)). By applying the definition of n_2 (page 15), we get:

$$\log^{(2)}(C_{36}) \leq 3\frac{(\beta+1)}{2} \log C_{33} \leq \frac{\beta+1}{2} n_2$$

Moreover,

$$\frac{\beta+1}{2}n_2 + \left(\frac{\beta+3/2}{\log(3/2)}\right)(\beta+2)n_2 + 2(\beta+3/2) \leq \frac{\beta+3/2}{\log(3/2)}(\beta+3)(n_2+2)$$

Therefore,

$$|Dh|_0 \leq e^{(3)} \wedge \left(\frac{2}{\log(3/2)}(\beta+3/2)(\beta+3)(n_2+2)(1+\log^{(2)}(|Df|_0))(1+\log^{(2)}(1/C_d))\right)$$

We have: $n_2 + 2 \leq 2n_2$. By lemma 4.22, and since $4/\log(3/2) \leq 10$, we get:

$$|Dh|_0 \leq e^{(3)} \wedge \left((\beta+3/2)(\beta+3)(k+4)!C_5(1+\log(1/C_d))^2(1+\log(\max(1,|Sf|_{k-3})))\right) \tag{39}$$

with $C_5 = 10(1+\log^{(2)}(|Df|_0))e^{(2)} \wedge (3W(f) + 2\log(\max(1,|Sf|_0))) + 2)$.

This estimation of $|Dh|_0$ is increasing with k. Therefore, to obtain a bound as low as we can, we take $k = \lceil 3\beta + 9/2 \rceil$. We obtain:

$$|Dh|_0 \leq e^{(3)} \wedge \left(C_3[\beta]C_4[C_d]C_5[|Df|_0, W(f), |Sf|_0]C_6[|Sf|_{\lceil 3\beta+3/2\rceil}]\right)$$

with:

1. $C_3[\beta] = (\lceil 3\beta + 21/2 \rceil)!$
2. $C_4[C_d] = (1+\log(1/C_d))^2$
3. $C_5[|Df|_0, W(f), |Sf|_0] = 10(1+\log^{(2)}(|Df|_0))e^{(2)} \wedge (3W(f) + 2\log(\max(1,|Sf|_0))) + 4)$
4. $C_6[|Sf|_{\lceil 3\beta+3/2\rceil}] = 1 + \log(\max(1, |Sf|_{\lceil 3\beta+3/2\rceil}))$

4.4 Proof of estimation (3)

Let $\delta = k - 2\beta - 1$ and $\beta > 0$. We make a Taylor expansion with $\delta \to 0$ (since $k \geq 3$, this implies automatically $\beta > 0$). To estimate $|Dh|_0$, we successively estimate n_2, n_3, $\rho/(1-\rho)$ and α'_{n_4}.

We have:

$$\theta = \frac{\delta}{4(3+\beta)} + o(\delta)$$

Since $\beta > 0$, then for δ sufficiently small, $C_{33} = e^{\frac{2}{1-(C_{19}^f)^\theta}}$. This makes the dependence on k and $|Sf|_{k-3}$ disappear. Moreover,

$$C_{33} = e \wedge \left(\frac{2}{\theta\log(1/C_{19}^f) + o(\theta)}\right)$$

$$n_2 = \left(\frac{4}{(\log C_{19}^f)^2} + \frac{2\log(1/C_d)}{\log 2}\right)\frac{1}{\theta} + o\left(\frac{1}{\theta}\right)$$

We denote $C_{40} = \frac{4}{(\log C_{19}^f)^2}$ and $C_{41} = \frac{2\log(1/C_d)}{\log 2}$.

Since

$$\beta_1 = \beta + \frac{C_{41}}{n_2} + o(\theta)$$

then

$$\log\left(\frac{\frac{k+1}{2} - \theta}{1+\beta_1}\right) = \frac{\frac{k+1}{2} - \theta}{1+\beta_1} - 1 + o(\theta) = \frac{\delta/2 - \theta - C_{41}/n_2}{1+\beta_1} + o(\theta)$$

Therefore,

$$n_3 = n_2(1 + \log(1+\beta))\left(\frac{1+\beta_1}{\delta/2 - \theta - C_{41}/n_2}\right) + o\left(\frac{1}{\theta^2}\right)$$

$$n_3 \le (1+\beta)^2 \frac{n_2^2}{\delta n_2/2 - \theta n_2 - C_{41}} + o\left(\frac{1}{\theta^2}\right)$$

Moreover, $\theta n_2 = C_{41} + C_{40} + o(1)$ and $\delta n_2 = 4(3+\beta)(C_{41} + C_{40}) + o(1)$. Therefore,

$$n_3 \le \frac{(1+\beta)^2(C_{41} + C_{40})}{\theta^2(4 + 2\beta)} + o\left(\frac{1}{\theta^2}\right) \tag{40}$$

Moreover,

$$\rho = \frac{\beta_1}{\frac{k-1}{2} - \theta} = \frac{\beta + \frac{C_{41}}{C_{41}+C_{40}}\theta + o(\theta)}{\beta + \delta/2 - \theta} = 1 - \frac{\theta}{\beta}\left(2(3+\beta) - 1 - \frac{C_{41}}{C_{41} + C_{40}}\right) + o(\theta)$$

Therefore, $\rho/(1-\rho) = o(1/\delta^2)$ (we recall that $1/\delta = o(1/\delta^2)$).
Let

$$C_7[k, C_d, |Df|_0, W(f), |Sf|_0] = \frac{(k+5)^2(k+1)^3}{2k\log 2}\left(\frac{2\log 2}{(\log C_{19}^f)^2} + \log(1/C_d)\right)$$

Since $k = 2\beta + 1 + \delta$, and by applying estimation (40), we have:

$$\alpha'_{n_4} \ge C_d \wedge \left(e \wedge \left(\frac{C_7}{\delta^2} + o\left(\frac{1}{\delta^2}\right)\right)\right)$$

Therefore,

$$|Dh|_0 \le C_{37}|Df|_0 \wedge \left((1/C_d) \wedge \left(e \wedge \left(\frac{1}{\delta^2}C_7 + o(1/\delta^2)\right)\right)\right)$$

$$|Dh|_0 \le C_{37}|Df|_0 \wedge \left(e \wedge \left(e \wedge \left(\log\log(1/C_d) + \frac{1}{\delta^2}C_7 + o(1/\delta^2)\right)\right)\right)$$

Since $\log^{(2)}(1/C_d) = o(1/\delta^2)$, then

$$|Dh|_0 \le C_{37}|Df|_0 \wedge \left(e \wedge \left(e \wedge \left(\frac{1}{\delta^2}C_7 + o(1/\delta^2)\right)\right)\right)$$

Likewise, since $|\log\log|Df|_0| \le e^{o(1/\delta^2)}$ and $|\log C_{37}| \le e \wedge e \wedge (o(1/\delta^2))$. We conclude:

$$|Dh|_0 \leq e^{(3)} \wedge \left(\frac{1}{\delta^2} C_7[k, C_d, W(f), |Sf|_0] + o(1/\delta^2)\right)$$

with:

$$C_7[k, C_d, W(f), |Sf|_0] = \frac{(k+5)^2(k+1)^3}{2k\log 2}\left(\frac{2\log 2}{(\log C_{19}^f)^2} + \log(1/C_d)\right)$$

We recall that:

$$C_{19}^f = \left(1 + e \wedge \left(-6\sqrt{2}e^{2W(f)}(|Sf|_0^{1/2} \vee 1)\right)\right)^{-1/2}$$

In estimations (2) and (3), three iterations of the exponential appear. This calls for explanation. A first exponential comes from the estimation $|Df^n|_0 \leq C|Df|_0^{2/\alpha_{n_4}}$, where n_4 is the rank above which the "favorable" case always occurs. A second exponential comes from writing $\alpha_{n_4} = \alpha_0^{\prod_{n=0}^{n_4-1}\left(\frac{1}{1-\eta_n}\right)}$. We bound each $\frac{1}{1-\eta_n}$ using the Diophantine condition, and a third exponential comes from the estimation $\prod_{n\in E_2}\left(\frac{1}{1-\eta_n}\right) \leq C^{n_3-n_2}$, where E_2 is the set and $n_3 - n_2$ is the number of "unfavorable" cases.

This number is bounded logarithmically, by $C\log C_{33}$. However, C_{33} is bounded by an exponential of the parameters. Indeed, when δ is small, $C_{33} \sim e^{\frac{1}{\delta}}$, which gives estimation (3). Otherwise, $C_{33} \sim C_{28}^{f,k}$. In this case, $C_{28}^{f,k} \sim C^k$. Indeed, in lemma 14, we need $k-1$ iterations to estimate $|D^{k-1}\log Df^p(x)|_0$ $(p \leq q_{n+1})$, an estimation that, in turn, gives an estimate of $C_{28}^{f,k}$. This gives estimation (2). Thus, we have explained the occurence of three exponentials in the estimates.

Since the number of "unfavorable" cases drives the dominant term of these estimates, they can be substantially improved when the "favorable" case always occurs. In remark 1.5, we make this assumption, together with the assumption $k \geq 3\beta + 9/2$. Thus, we can take $\theta = 1/2$, and a sufficient condition for the occurrence of the "favorable" case is:

$$\frac{\alpha_{n+1}}{\alpha_n} \geq C_{28}^{f,k}(C_{19}^f)^{(n-1)\frac{k}{2}} = C_8(n, k, \beta, W(f), |Sf|_{k-3})$$

which decreases geometrically with n.
We recall that:

$$C_{19}^f = \left(1 + e \wedge \left(-6\sqrt{2}e^{2W(f)}(|Sf|_0^{1/2} \vee 1)\right)\right)^{-1/2}$$

$$C_{28}^{f,k} = (k+3)^{(k+3)!}e^{(k+3)!W(f)}(\max(1, |Sf|_{k-3}))^{(k+1)!}$$

We obtain the following estimation:

$$|Dh|_0 \leq \exp\left(C_9[k, W(f), |Sf|_{k-3}]^{C_{10}(\beta)}\right)|Df|_0^2$$

with:

$$C_9[k, W(f), |Sf|_{k-3}] = \max\left(e^{\frac{2}{1-(C_{19}^f)^{1/2}}}, 4C_{28}^{f,k}\right)$$

$$C_{10}[\beta] = \frac{3\beta + 1}{2}$$

Finally, note that numbers of constant type do not always satisfy (4) for any n (they only satisfy it above some rank). Moreover, there are numbers satisfying (4) that are not of constant type.

5 C^k estimations

In this section, we compute estimates of higher order derivatives of the conjugacy h in function of bounds on the first derivative of h. We compute the values of some of the constants appearing in Yoccoz's proof [14] (we do not compute the dependency in k). However, in order to obtain our result, we need to slightly modify the proof of one proposition (proposition 5.10). If we strictly followed Yoccoz's proof, we would find an estimate that depends on the C^1-norm of h, and on $k, \beta, C_d, W(f), |Sf|_{k-3}, |D^{k-1} \log Df|_0$, but this estimate would diverge as f gets closer to a rotation. Moreover, we need to elaborate on the end of his proof.

The proof has four steps. We let real numbers $0 \leq \gamma_0 < \gamma_1 < g(\gamma_0)$, with $g(\gamma_0) = ((1+\beta)\gamma_0 + k - (2+\beta)) \,/\, (2+\beta)$, and we let an integer N. In the first three steps, we compute $\| \log Df^N\|_{\gamma_1}$ in function of $\sup_{p\geq 0} \| \log Df^p\|_{\gamma_0}$ (estimation (41)). In the first step, using convexity estimations (proposition 5.7) and a consequence of the Faa-di-Bruno formula (lemma 5.6), we establish an estimation of $\| \log Df^{q_s}\|_\gamma$ for $0 \leq \gamma \leq k-1$ (lemma 5.8).

In the second step, we obtain an estimation of $\| \log Df^{nq_s}\|_\gamma$, $0 \leq n \leq q_{s+1}/q_s$ for $0 \leq \gamma \leq \gamma_1$ (estimation (54)).

In the third step, we write $N = \sum_{s=0}^{S} b_s q_s$, with b_s integers satisfying $0 \leq b_s \leq q_{s+1}/q_s$, in order to get an estimation of $\| \log Df^N\|_{\gamma_1}$ in function of $\sup_{p\geq 0} \| \log Df^p\|_{\gamma_0}$. Thus, in these three steps, the aim is to establish the following proposition:

Proposition 5.1. *Let* $0 \leq \gamma_0 < \gamma_1 < g(\gamma_0) = \frac{(1+\beta)\gamma_0 + k - (2+\beta)}{2+\beta}$*. We have:*

$$\| \log Df^N\|_{\gamma_1} \leq e \wedge \left(C_{72}(k,\beta) \left(C_{73}^{f,k} + 4k^2 \log(C_d^{-1}) + 24k(k+1) \sup_{p\geq 0} \| \log Df^p\|_{\gamma_0} \right)^4 \right) \tag{41}$$

The expressions of C_{72} and $C_{73}^{f,k}$ are given page 45.

In the fourth step, we iterate this reasoning: the inductive step is given by proposition 5.1: if we have an estimate of $\sup_{N\geq 0} \| \log Df^N\|_{\gamma_i}$, then we can get an estimate of $\sup_{N\geq 0} \| \log Df^N\|_{\gamma_{i+1}}$ for $\gamma_i < \gamma_{i+1} < g(\gamma_i)$. We can initiate the induction with $\gamma_0 = 0$, because we have C^1 estimates. We take $\gamma_{i+1} = \frac{1}{2}(g(\gamma_i) + \gamma_i)$ and we have: $\lim_{i\to+\infty} \gamma_i = k - 2 - \beta$. Thus, we can obtain an estimation of $\|Dh\|_{k-2-\beta-\eta}$.

In all the rest of the paper, we denote:

$$M' = \exp\left(\sup_{i\geq 0} | \log D(f^i)|_0 \right)$$

$$M = \exp\left(\sup_{i\geq 0} \|\log D(f^i)\|_{\gamma_0}\right)$$

Note that $M \geq M' \geq 1$.

5.1 Estimation of $\|\log Df^{q_s}\|_\gamma$, $0 \leq \gamma \leq k-1$.

The following lemma is a converse of the implication used in [14, p. 348], according to which if M_n/m_n is bounded, then the conjugacy of f to a rotation is C^1:

Lemma 5.2. *Let $M_n = \sup_{x\in\mathbb{T}^1} |f^{q_n}(x) - x|$, $m_n = \inf_{x\in\mathbb{T}^1} |f^{q_n}(x) - x|$ and $M' = \exp\left(\sup_{i\geq 0} |\log D(f^i)|_0\right)$. Then we have the following estimation:*

$$\frac{M_n}{m_n} \leq M'$$

Proof. Let $\epsilon > 0$, x, y such that $M_n = |f^{q_n}(x) - x|$ and $m_n = |f^{q_n}(y) - y|$.
Since $f^p(y)_{p\geq 0}$ is dense in $\mathbb{T}^1$, then there is a positive integer l such that $|f^l(y) - x| \leq \min\left(\frac{\epsilon}{|Df^{q_n}|_0}, \epsilon\right)$.
Then we obtain:

$$|f^{q_n}(x)-x| \leq |f^{q_n}(x)-f^{q_n}(f^l(y))|+|f^l(f^{q_n}(y))-f^l(y)|+|f^l(y)-x| \leq |Df^l|_0|f^{q_n}(y)-y|+2\epsilon \leq M'm_n+2\epsilon$$

for all $\epsilon > 0$. Hence the lemma.

□

The C^γ-norms, when γ varies in $\mathbb{R}^+$, are related with each other by convexity inequalities (also called interpolation inequalities):

Proposition 5.3. *Let $\gamma_2, \gamma_3 \in \mathbb{R}^+$ with $0 \leq \gamma_2 \leq \gamma_3$ and $\gamma_3 > 0$. For any $\phi \in C^{\gamma_3}(\mathbb{T}^1)$, we have:*

$$\|\phi\|_{\gamma_2} \leq C_{42}(\gamma_3)\, \|\phi\|_0^{\frac{\gamma_3-\gamma_2}{\gamma_3}}\, \|\phi\|_{\gamma_3}^{\frac{\gamma_2}{\gamma_3}}$$

with $C_{42}(\gamma_3) = 2^{(\lfloor\gamma_3\rfloor+1)^2+1}$.

Proof. See appendix. □

Using these convexity inequalities, we establish various relations, among which is the important corollary 5.5, which relate the norms of $\log Df^n$ and those of $Df^n - 1$.

For $0 \leq \gamma_1 \leq \gamma_2 \leq k-1$, $\gamma_2 \neq 0$, and $n \in \mathbb{Z}$, proposition 5.3 gives:

$$\|\log Df^n\|_{\gamma_1} \leq C_{43}\|\log Df^n\|_{\gamma_2}^{\gamma_1/\gamma_2} \tag{42}$$

$$\|Df^n - 1\|_{\gamma_1} \leq C_{44}\|Df^n - 1\|_{\gamma_2}^{\gamma_1/\gamma_2} \tag{43}$$

with $C_{43}(\gamma_1, \gamma_2) = C_{42}(\gamma_2)(\log M')^{1-\gamma_1/\gamma_2}$ and $C_{44}(\gamma_1, \gamma_2) = C_{42}(\gamma_2)(M'+1)^{1-\gamma_1/\gamma_2}$.

For $n \geq 0$, $j \in \mathbb{Z}$, we have:

$$||(Df^n)^j||_0 \leq M'^{|j|} \tag{44}$$

and, for $0 < \gamma' < 1$, since $|Df^n - 1|_{\gamma'} = |Df^n|_{\gamma'}$:

$$|(Df^n)^j|_{\gamma'} \leq |j| M'^{|j|-1} |Df^n - 1|_{\gamma'} \tag{45}$$

Therefore, for $0 \leq \gamma' < 1$, $\phi \in C^{\gamma'}(\mathbb{T}^1)$, we get, by the relations (8) and (9):

$$||(Df^n)^j \phi||_{\gamma'} \leq C_{45}(||\phi||_{\gamma'} + ||Df^n - 1||_{\gamma'} ||\phi||_0) \tag{46}$$

with $C_{45}(j) = (|j| + 1)M'^{|j|}$.

Let $\Delta = X_1^{j_1}...X_l^{j_l}$ be a monomial of l variables, such that $l = \sum_{p=1}^{l} p j_p \geq 1$. Let $0 \leq \gamma' < 1$, $n \in \mathbb{Z}$. We estimate $||\Delta||_{\gamma'}$ when $X_i = D^i \log Df^n$ or when $X_i = D^{i+1} f^n$, supposing that $l + \gamma' \leq k - 1$.

The relations (8) and (9) allow estimating $||\Delta||_{\gamma'}$ by a sum of less than 2^l terms of the form $||X_p||_{\gamma'} ||\Delta / X_p||_0$, $1 \leq p \leq l$, $j_p \neq 0$. By relation (42), we have:

$$\begin{aligned}
||D^p \log Df^n||_{\gamma'} &\leq C_{46} || \log Df^n||_{l+\gamma'}^{(p+\gamma')/(l+\gamma')} \\
||D^{p+1} f^n||_{\gamma'} &\leq C_{47} ||Df^n - 1||_{l+\gamma'}^{(p+\gamma')/(l+\gamma')} \\
\left\| \frac{\Delta(D \log Df^n, ..., D^l \log Df^n)}{D^p \log Df^n} \right\|_0 &\leq C_{48} || \log Df^n||_{l+\gamma'}^{(l-p)/(l+\gamma')} \\
\left\| \frac{\Delta(D^2 f^n, ..., D^{l+1} f^n)}{D^{p+1} f^n} \right\|_0 &\leq C_{49} ||Df^n - 1||_{l+\gamma'}^{(l-p)/(l+\gamma')}
\end{aligned}$$

with $C_{46}(p, l, \gamma') = C_{43}(p + \gamma', l + \gamma')$; $C_{47}(p, l, \gamma') = C_{44}(p + \gamma', l + \gamma')$;
$C_{48}(p, l, \gamma') = (C_{43}(1, l + \gamma'))^{l-1}$;
$C_{49}(p, l, \gamma') = (C_{44}(1, l + \gamma'))^{l-1}$

Scheme of the proof. The first two estimates are straightforward. For the third estimate, we write

$$\frac{\Delta(D \log Df^n, ..., D^l \log Df^n)}{D^p \log Df^n} = (D \log Df^n)^{j_1} ... (D^p \log Df^n)^{j_p - 1} ... \left(D^l \log Df^n\right)^{j_l}$$

we apply estimation (42) to each $D^i \log Df^n$ and we use that $\sum j_k \leq l$.
The proof of the fourth estimate is analogous, by noting that for $i \geq 1$,
$D^i(Df^n - 1) = D^{i+1} f^n$.

□

Therefore, when $X_i = D^i \log Df^n$, we get:

$$||\Delta||_{\gamma'} \leq C_{50} || \log Df^n||_{l+\gamma'} \tag{47}$$

and when $X_i = D^{i+1} f^n$,

$$||\Delta||_{\gamma'} \leq C_{51} ||Df^n - 1||_{l+\gamma'} \tag{48}$$

with $C_{50}(l) = 2^l \max_{1\le p\le l} \sup_{0\le\gamma'<1} C_{46}(p,l,\gamma')C_{48}(p,l,\gamma')$ and
$C_{51}(l) = 2^l \max_{1\le p\le l} \sup_{0\le\gamma'<1} C_{47}(p+\gamma, l+\gamma')C_{49}(p+\gamma', l+\gamma')$.

Using appendix 6.4, this allows obtaining the following lemma:

Lemma 5.4. *Let P be one of the polynomials of appendix 6.4. P is a polynomial of l variables $X_1, ..., X_l$, homogeneous of weight l if X_i has a weight of i. For all $n \in \mathbb{Z}$, all $0 \le \gamma' < 1$, we have:*

$$\begin{aligned} \|P(D\log Df^n, ..., D^l \log Df^n)\|_{\gamma'} &\le C_{52}\|\log Df^n\|_{l+\gamma'} \\ \left\| P\left(\frac{D^2 f^n}{Df^n}, ..., \frac{D^{l+1} f^n}{Df^n}\right)\right\|_{\gamma'} &\le C_{53}\|Df^n - 1\|_{l+\gamma'} \end{aligned}$$

with $C_{52}(l) = (4(l+1))^{4(l+1)}C_{50}(l)$ and
$C_{53}(l) = (4(l+1))^{4(l+1)}C_{45}(-l)C_{51}(l)\,(1 + C_{44}(l, l+\gamma')C_{44}(\gamma', l+\gamma'))$

Scheme of the proof. The first estimate comes from the preceding discussion. For the second estimate, we write a monomial of P as:

$$\left(\frac{D^2 f^n}{Df^n}\right)^{j_1} ... \left(\frac{D^l f^n}{Df^n}\right)^{j_l} = (D^2 f^n)^{j_1} ... (D^l f^n)^{j_l} (Df^n)^{-\sum j_k}$$

We apply estimate (46), (48), and estimate (43) twice.

□

Corollary 5.5. *For $n \in \mathbb{Z}$, $0 \le \gamma \le k-1$, $\gamma = \lfloor\gamma\rfloor + \gamma'$, $0 \le \gamma' < 1$. we have:*

$$C_{54}^{-1}\|Df^n - 1\|_\gamma \le \|\log Df^n\|_\gamma \le C_{54}\|Df^n - 1\|_\gamma$$

with $C_{54}(\gamma) = M$ if $0 \le \gamma < 1$ and
$C_{54}(\gamma) = [C_{52}(\lfloor\gamma\rfloor)C_{45}(1)(1 + MC_{43}(\lfloor\gamma\rfloor.\gamma)C_{43}(\gamma',\gamma))] \vee C_{53}(\lfloor\gamma\rfloor)$ if $\gamma \ge 1$.

Scheme of the proof. For $0 \le \gamma < 1$, we prove the estimates directly, using that $\log x \le x - 1$.

When $\gamma \ge 1$, for the right-hand side of the estimation, we use formula (77) in appendix 6.4 and the second estimate of lemma 5.4.

For the left-hand side, we apply formula (76) in appendix 6.4, the first estimate of lemma 5.4, relation (46) with $\phi = D^{\lfloor\gamma\rfloor+1} f^n / Df^n$ and $j = 1$, the left-hand side of this estimate of corollary 5.5 with $\gamma < 1$, and relation (42) twice.

□

Using mainly the Faa-d-Bruno formula, we have the lemma [14, p. 350]:

Lemma 5.6. *Let $\gamma_0 \ge \gamma \ge 0$, $\psi \in D^{\max(1,\gamma)}(\mathbb{T}^1)$, $\phi \in C^\gamma(\mathbb{T}^1)$. We have:*

$$\|\phi \circ \psi\|_\gamma \le C_{55}\|\phi\|_\gamma$$

with $C_{55}(\gamma,\psi) = \|\psi\|_\gamma$ for $0 \le \gamma < 1$, and $C_{55}(\gamma,\psi) = ((2\lfloor\gamma\rfloor)!)^2 \|D\psi\|_{\gamma-1}^{\lfloor\gamma\rfloor+1}$ for $\gamma \ge 1$.

When $\psi = f^n$ for some integer n, or when $\psi = h^{-1}$, where h is the conjugacy of f to a rotation, we note:
$C_{55}(\gamma, \psi) = C_{55}(\gamma) = M^{\gamma}$ if $0 \le \gamma < 1$, and
$C_{55}(\gamma, \psi) = C_{55}(\gamma) = ((2\lfloor\gamma\rfloor)!)^2 M^{\lfloor\gamma\rfloor+1}$ if $\gamma \ge 1$.

Proof. See appendix 6.3. □

We have: $f^n = hR_{n\alpha}h^{-1}$. We apply lemma 5.6 with $\psi = h^{-1}$ and $\phi = hR_{n\alpha} - h - n\alpha$. To estimate $\|hR_{n\alpha} - h - n\alpha\|_\gamma$, we use the C^γ-norm of Dh. We get:

Proposition 5.7. *For $n \in \mathbb{Z}$, $0 \le \gamma \le \gamma_0$, we have:*

$$\|f^n - id - n\alpha\|_\gamma \le C_{56}\|n\alpha\|$$

with $C_{56}(\gamma) = 2MC_{55}(\gamma)$.

Let $\alpha_n = (-1)^n(q_n\alpha - p_n)$ and let $\Delta_s = \|D^{k-1}\log Df^{q_s}\|_0 + \alpha_s$ (the role of this additional α_s is explained at the end of the proof of lemma 5.12). We could also have taken $\Delta_s = \|D^{k-1}\log Df^{q_s}\|_0 \vee \alpha_s)$. By applying lemma 4.4, and since $M_{s-1} \le M'\alpha_{s-1}$, $1/m_{s-1} \le M'/\alpha_{s-1}$, and $\alpha_{s-1} \le 1/q_s$, then we have:

$$\Delta_s \le (C_{27}^{f,k} M'^{\frac{3}{2}(k-1)} + 1) q_s^{(k-1)/2} \tag{49}$$

Using corollary 5.5, convexity inequalities (proposition 5.3), proposition 5.7, and corollary 5.5 again, we obtain the following lemma:

Lemma 5.8. *Let $\gamma \in [0, k-1]$ and $s \ge 0$. We have:*

$$\|\log Df^{q_s}\|_\gamma \le C_{57}(\gamma, \gamma_0) q_{s+1}^{-1} (q_{s+1}\Delta_s)^{\max(0, \frac{\gamma+1-\gamma_0}{k-\gamma_0})}$$

with $C_{57}(\gamma, \gamma_0) = C_{42}(k) C_{54}(\gamma)^{1+\max(0, \frac{\gamma+1-\gamma_0}{k-\gamma_0})} C_{56}(\gamma_0)^{1-\max(0, \frac{\gamma+1-\gamma_0}{k-\gamma_0})}$.

We make a remark on the method and notation: in this lemma 5.8, we estimate the C^γ-norm for $0 \le \gamma \le k-1$, instead of only estimating the C^{γ_1}-norm, because of two reasons: first, this lemma is used to obtain lemma 5.9, in which we need an estimation of all the norms of order $\gamma \le k-1$. Second, in the proof of proposition 5.10, we need an estimate of $\|\log Df^{q_s}\|_{k-1}$.

5.2 Estimation of $\|\log Df^{nq_s}\|_\gamma$, $0 \le n \le q_{s+1}/q_s$, $0 \le \gamma \le k-1$.

We use lemma 5.8 to estimate $\|\log Df^{nq_s}\|_\gamma$, $0 \le \gamma \le k-1$ (lemma 5.9) and second, we bootstrap this estimate (lemma 5.12). This bootstrapping allows getting a higher degree of differentiability γ_1 at the end (see estimation (55)).

The Diophantine condition on α implies $q_{s+1} \le C_d^{-1} q_s^{1+\beta}$. Therefore, by applying estimation (49), we get:

$$(\Delta_s q_{s+1})^{1/k} q_s^{-1} \le C_{58}(0) q_s^{-\epsilon} \tag{50}$$

With $\epsilon = \frac{1}{2} - \frac{1+2\beta}{2k} > 0$ and $C_{58}(0) = \left[(C_{27}^{f,k} M^{\frac{3}{2}(k-1)} + 1) C_d^{-1}\right]^{1/k}$

The preceding estimates give the lemma:

Lemma 5.9. *Let $\gamma \in [0, k-1]$. For $s \geq 0$, $0 \leq n \leq q_{s+1}/q_s$, we have:*

$$\|\log Df^{nq_s}\|_\gamma \leq C_{59}(\gamma, 0) q_s^{-1}(q_{s+1}\Delta_s)^{(\gamma+1)/k}$$

where, if $0 \leq \gamma < 1$, $C_{59}(\gamma, 0) = C_{57}(\gamma, 0)$, and if $\gamma \geq 1$, $C_{59}(\gamma, 0)$ is defined inductively by:

$$C_{59}(\gamma, 0) = C_{59}(\gamma - 1, 0)\lfloor\gamma\rfloor^2 C_{58}(0) C_{52}(\lfloor\gamma\rfloor - 1) C_{57}(\gamma, 0) M^\gamma \left[2 + C_{43}(\gamma', \gamma) C_{43}(\lfloor\gamma\rfloor, \gamma)\right]$$

Scheme of the proof. This lemma is shown by induction on $r = \lfloor\gamma\rfloor$. If $r = 0$, we write $\log Df^{nq_s} = \sum_{i=0}^{n-1} \log Df^{q_s} \circ f^i$ and we apply lemma 5.8.

Suppose the lemma holds for $r - 1 + \gamma'$, with $0 \leq \gamma' < 1$. We have, using the expression (79) in appendix 6.4, and using estimations (8) and (9):

$$\|D^r \log Df^{nq_s}\|_{\gamma'} \leq \sum_{l=0}^{r-1} \sum_{i=0}^{n-1} (A_{i,l} + B_{i,l} + C_{i,l}) \tag{51}$$

with:

$$A_{i,l} = \|D^{r-l} \log Df^{q_s} \circ f^{iq_s}\|_{\gamma'} \|(Df^{iq_s})^{r-l}\|_0 \|E_l^r\|_0$$

$$B_{i,l} = \|D^{r-l} \log Df^{q_s} \circ f^{iq_s}\|_0 |(Df^{iq_s})^{r-l}|_{\gamma'} \|E_l^r\|_0$$

$$C_{i,l} = \|D^{r-l} \log Df^{q_s} \circ f^{iq_s}\|_0 \|(Df^{iq_s})^{r-l}\|_0 \|E_l^r\|_{\gamma'}$$

$$E_l^r = E_l^r(D \log Df^{iq_s}, ..., D^l \log Df^{iq_s})$$

We estimate E_l^r with lemma 5.4 (with the polynomial $P = E_l^r$), with (42) (for $B_{i,l}$) and with the induction assumption. We estimate $\|D^{r-l} \log Df^{q_s} \circ f^{iq_s}\|_{\tilde{\gamma}}$, $\tilde{\gamma} = 0$ or γ', by applying lemma 5.6 with $\phi = D^{r-l} \log Df^{q_s}$ and $\psi = f^{iq_s}$, and by applying lemma 5.8. We estimate $|(Df^{iq_s})^{r-l}|_0$ with (44). For $|(Df^{iq_s})^{r-l}|_{\gamma'}$, we apply (45), corollary 5.5, (42) and the induction assumption. We get:

$$A_{i,l} \leq C_{59}(l, 0) C_{52}(l) C_{57}(\gamma' + r - l, 0) M^{\gamma' + r - l} q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{\frac{r-l+\gamma'+1}{k} + \frac{l+1}{k}}$$

$$B_{i,l} \leq C_{57}(r-l, 0) C_{52}(l)(r-l) M^{r-l} C_{43}(l, l+\gamma') C_{43}(\gamma', l+\gamma') C_{59}(l+\gamma', 0) q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{\frac{r-l+1}{k} + \frac{l+\gamma'+1}{k}}$$

$$C_{i,l} \leq M^{r-l} C_{52}(l) C_{57}(r-l, 0) C_{59}(l+\gamma', 0) q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{\frac{r-l+1}{k} + \frac{l+\gamma'+1}{k}}$$

Thus, we have:

$$A_{i,l} + B_{i,l} + C_{i,l} \leq C_{60}(l, \gamma', 0) q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{\frac{\gamma+1}{k} + 1/k}$$

with:

$$C_{60}(l, \gamma', 0) = C_{59}(l+\gamma', 0) C_{52}(l) C_{57}(r-l+\gamma', 0) M^{r-l} \left[M^{\gamma'} + (r-l) C_{43}(\gamma', l+\gamma') C_{43}(l, l+\gamma') + 1\right]$$

We conclude using estimation (50), and using the fact that the sum (51) has $rn \leq \lfloor\gamma\rfloor q_{s+1}/q_s$ terms.

□

By applying this lemma 5.9, together with estimate (50), lemma 5.8 and lemma 5.4, we get the proposition [14, p.355]:

Proposition 5.10. *The sequence* $(\Delta_s/q_s)_{s\geq 0}$ *is bounded by* C_{61}.

C_{61} is defined by the following:

$$\begin{aligned} C_{62} &= (k-2)M^{k-1}C_{52}(k-2)C_{59}(k-2,0)C_{58}(0)C_{57}(k-1,0); \\ C_{61} &= 2C_{27}^{f,k}M^{k-1}\prod_{s=0}^{\infty}\left(1+\frac{5C_{62}}{q_s^{\epsilon}}\right) \end{aligned}$$

Proof. We slightly modify Yoccoz's proof. Let $\Delta'_{-1} = 0$ and, for $s \geq 0$:

$$\Delta'_s = \sup\{|D^{k-1}\log Df^{q_t}\circ f^m(Df^m)^{k-1}|_0, 0 \leq t \leq s, m \geq 0\}$$

For $s \geq 0$, we have: $\Delta_s \leq \Delta'_s + \alpha_s$ (This implies $\Delta_s \leq C\Delta'_s$ when f is not a rotation, but contrary to Yoccoz's proof, we do not use this estimate, because the constant C is of the form $C = 1 + \frac{M^{k-1}}{|D^{k-1}\log Df|_0}$, which diverges as f gets closer to a rotation). We compute a bound on $(\Delta'_s + \alpha_s)/q_s$.

Let $s \geq 0$ (this is another difference with Yoccoz's proof, which only considers $s \geq 1$). We have: $q_{s+1} = a_{s+1}q_s + q_{s-1}$ (we recall that $q_{-1} = 0$). Using formula (78) in appendix 6.4 with $g = f^{q_{s-1}}$ and $h = f^{a_{s+1}q_s}$, we can write:

$$(D^{k-1}\log Df^{q_{s+1}}\circ f^m)(Df^m)^{k-1} = X' + Y' + Z'$$

with:

$$X' = (D^{k-1}\log Df^{q_{s-1}}\circ f^{a_{s+1}q_s+m})(Df^{a_{s+1}q_s}\circ f^m)^{k-1}(Df^m)^{k-1}$$

$$Y' = D^{k-1}\log Df^{a_{s+1}q_s}\circ f^m(Df^m)^{k-1}$$

$$Z' = \sum_{l=1}^{k-2}(D^{k-1-l}\log Df^{q_{s-1}}\circ f^{a_{s+1}q_s+m})(Df^{a_{s+1}q_s}\circ f^m)^{k-1-l}(Df^m)^{k-1}G_l^{k-1}(D\log Df^{a_{s+1}q_s}\circ f^m, ..., D^l\log Df^{a_{s+1}q_s}\circ f^m)$$

We have:

$$|X'|_0 \leq \Delta'_{s-1}$$

Using formula (79) in appendix 6.4 with $g = f^{q_s}$, we have:

$$Y' = \sum_{l=0}^{k-2}\sum_{n=0}^{a_{s+1}-1}(D^{k-1-l}\log Df^{q_s}\circ f^{nq_s+m})(Df^{nq_s+m})^{k-1-l}E_l^{k-1}(D\log Df^{nq_s}, ..., D^l\log Df^{nq_s})\circ f^m(Df^m)^l = \sum_{l=0}^{k-2}Y'_l$$

(with the convention $E_0^{k-1} = 1$). We have: $|Y'_0|_0 \leq a_{s+1}\Delta'_s$.

For $l \geq 1$, we estimate $E_l^{k-1}(D\log Df^{nq_s}, ..., D^l\log Df^{nq_s})\circ f^m(Df^m)^l$ using lemma 5.6 (with $\psi = f^m$ and $\gamma = 0$), lemma 5.4 (with $P = E_l^{k-1}$), lemma 5.9 (with $\gamma = l$) and estimation (50). We get:

$$|E_l^{k-1}(D\log Df^{nq_s}, ..., D^l\log Df^{nq_s})\circ f^m(Df^m)^l|_0 \leq M^lC_{52}(l)C_{59}(l,0)C_{58}(0)(\Delta_sq_{s+1})^{l/k}q_s^{-\epsilon}$$

By applying lemma 5.8 (with $\gamma = k-1-l$ and $\gamma_0 = 0$), and using that $\Delta_s \leq \Delta'_s + \alpha_s$, we get:

$$|Y'_l|_0 \leq a_{s+1}(\Delta'_s + \alpha_s)M^{k-1}C_{57}(k-1-l, 0)C_{52}(l)C_{59}(l, 0)C_{58}(0)q_s^{-\epsilon}$$

Therefore,

$$|Y'|_0 \leq a_{s+1}\Delta'_s + a_{s+1}(\Delta'_s + \alpha_s)C_{62}q_s^{-\epsilon}$$

Likewise, we can show that, for $s \geq 1$:

$$|Z'|_0 \leq C_{62}q_s^{-\epsilon}q_s^{-1}\left(q_s\Delta_{s-1}\right)^{\frac{k-l}{k}}\left(q_{s+1}\Delta_s\right)^{l/k}$$

(Yoccoz concludes the estimation of $|Z'|_0$ here, using the fact that $q_s^{1-l/k} \leq q_{s+1}^{1-l/k}$ and using the fact that $\Delta_t \leq C\Delta'_t$, $t = s-1, s$. We don't use these facts.)

Since $\Delta_t \leq \Delta'_t + \alpha_t$, $t = s-1, s$, we get:

$$|Z'|_0 \leq C_{62}q_s^{-\epsilon}\left(\frac{q_{s+1}}{q_s}\right)^{l/k}\left(\Delta'_{s-1} + \alpha_{s-1}\right)^{1-l/k}\left(\Delta'_s + \alpha_s\right)^{l/k}$$

$$|Z'|_0 \leq C_{62}q_s^{-\epsilon}\frac{q_{s+1}}{q_s}\left(\Delta'_s + \alpha_s\right)\left(\left(1 + \frac{\alpha_{s-1} - \alpha_s}{\Delta'_s + \alpha_s}\right)\left(\frac{q_s}{q_{s+1}}\right)\right)^{1-l/k}$$

Since $\Delta'_s \geq 0$, and since $a_{s+1} \leq q_{s+1}/q_s \leq 2a_{s+1}$ and $\alpha_{s-1} \leq 2a_{s+1}\alpha_s$, we get:

$$|Z'|_0 \leq 4C_{62}q_s^{-\epsilon}a_{s+1}\left(\Delta'_s + \alpha_s\right)$$

If $s = 0$, $Z' = 0$. This estimate still holds.

Therefore, for $s \geq 0$,

$$\alpha_{s+1} + |(D^{k-1}\log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \leq \alpha_{s+1} + \Delta'_{s-1} + a_{s+1}\Delta'_s + a_{s+1}(\Delta'_s + \alpha_s)5C_{62}q_s^{-\epsilon} \tag{52}$$

$$\alpha_{s+1} + |(D^{k-1}\log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \leq \alpha_{s+1} - a_{s+1}\alpha_s + \Delta'_{s-1} + a_{s+1}(\Delta'_s + \alpha_s)\left(1 + 5C_{62}q_s^{-\epsilon}\right)$$

Moreover, we have: $\alpha_{s-1} = a_{s+1}\alpha_s + \alpha_{s+1}$. Therefore, for $s \geq 1$, since $\alpha_{s+1} < \frac{1}{2}\alpha_{s-1}$, then

$$\alpha_{s+1} - a_{s+1}\alpha_s = 2\alpha_{s+1} - \alpha_{s-1} < 0 \leq \alpha_{s-1}$$

Therefore,

$$\alpha_{s+1} + |(D^{k-1}\log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0 \leq \max_{t=s-1,s}\frac{\alpha_t + \Delta'_t}{q_t}(q_{s-1} + a_{s+1}q_s)\left(1 + 5C_{62}q_s^{-\epsilon}\right)$$

Since $q_{s-1} + a_{s+1}q_s = q_{s+1}$, we get:

$$\frac{\alpha_{s+1} + |(D^{k-1}\log Df^{q_{s+1}} \circ f^m)(Df^m)^{k-1}|_0}{q_{s+1}} \leq \max_{t=s-1,s}\frac{\alpha_t + \Delta'_t}{q_t}\left(1 + 5C_{62}q_s^{-\epsilon}\right)$$

If $s = 0$, we have:

$$\frac{\alpha_1 + \Delta_1'}{q_1} \leq \frac{\alpha_0 + \Delta_0'}{q_0} (1 + 5C_{62})$$

Let $\theta_s = \max_{0\leq t\leq s} \frac{\alpha_t + \Delta_t'}{q_t}$. The preceding estimates give:

$$\theta_{s+1} \leq \theta_s (1 + 5C_{62} q_s^{-\epsilon})$$

Moreover,

$$\left(\frac{\Delta_0' + \alpha_0}{q_0}\right) \leq 1 + M^{k-1} |D^{k-1} \log Df|_0$$

Therefore, for any $s \geq 0$,

$$\frac{\Delta_s}{q_s} \leq \left(1 + M^{k-1} |D^{k-1} \log Df|_0\right) \prod_{s=0}^{+\infty} (1 + 5C_{62} q_s^{-\epsilon})$$

To conclude, we apply the claim:

Claim 5.11. *Let $C_{27}^{f,k}$ defined in lemma 4.4. For any $k \geq 3$, we have:*

$$|D^{k-1} \log Df|_0 \leq \tilde{C}_{27}(k, |Sf|_{k-3}) \leq C_{27}^{f,k}$$

Proof. First, we recall the observation (see e.g. [14]) that if x_0 is a point where $(D \log Df)^2$ is maximal, then we have:

$$|Sf|_0 \geq |Sf(x_0)| = \left|D^2 \log Df(x_0) - \frac{1}{2}(D \log Df(x_0))^2\right| = \left|\frac{1}{2}(D \log Df(x_0))^2\right| = \left|\frac{1}{2}(D \log Df)^2\right|_0$$

To prove the claim, we proceed by induction on k, using the fact that $|D \log Df|_0 \leq \sqrt{2|Sf|_0}$ and by applying formula (70) in appendix 6.1.
If $k = 3$,

$$|D^2 \log Df|_0 \leq |Sf|_0 + \frac{1}{2}|(D \log Df)^2|_0 \leq 2|Sf|_0$$

Suppose the estimate holds for all $r \leq k$. By formula (70), we have:

$$D^k \log Df = D^{k-2} Sf - G_k(D \log Df, ..., D^{k-1} \log Df)$$

As in the proof of lemma 4.4 (see appendix 6.1), we have:

$$\left|G_k(D \log Df, ..., D^{k-1} \log Df)\right| \leq \frac{(2(k-1))!}{2k} \left(\tilde{C}_{27}(k, |Sf|_{k-3})\right)^k$$

We conclude as in the proof of lemma 4.4.

Let us make a remark: by using lemma 5.4, we can improve this estimate. However, lemma 5.4 cannot be used to improve the estimate of lemma 4.4, because it is a pointwise estimate: an estimate of $|D^{k-1} \log Df^{q_s}(x)|$ in function of $M_n^{1/2}/m_n(x)$. If we only needed an estimate of $|D^{k-1} \log Df^{q_s}|_0$ in function of $M_n^{1/2}/m_n$, this improvement would be possible.

□

$\square$

With proposition 5.10, and by using the Diophantine condition $q_{s+1} \le C_d^{-1} q_s^{1+\beta}$, we can generalize estimation (50) and lemma 5.9, for $\gamma_0 > 0$. The generalization of estimation (50) is:

$$(\Delta_s q_{s+1})^{1/(k-\gamma_0)} q_s^{-1} \le C_{58}(\gamma_0) q_s^{\frac{\beta+2}{k-\gamma_0}-1} \tag{53}$$

with $C_{58}(\gamma_0) = C_{61}^{\frac{1}{k-\gamma_0}} C_d^{\frac{-1}{k-\gamma_0}}$.

The generalization of lemma 5.9 is:

Lemma 5.12. *Let $\gamma \in [0, k-1]$. For $s \ge 0$, $0 \le n \le q_{s+1}/q_s$, we have:*

$$\| \log Df^{nq_s} \|_\gamma \le C_{59}(\gamma, \gamma_0) q_s^{-1} (q_{s+1}\Delta_s)^{\frac{\gamma+1-\gamma_0}{k-\gamma_0} \vee 0}$$

where, if $0 \le \gamma < 1$, $C_{59}(\gamma, \gamma_0) = \max(C_{54}(\gamma)C_{56}(\gamma+1), C_{57}(\gamma, \gamma_0))$, and if $\gamma \ge 1$, $C_{59}(\gamma, 0)$ is defined inductively by:

$$C_{59}(\gamma, \gamma_0) = C_{59}(\gamma-1, \gamma_0)\lfloor\gamma\rfloor C_{58}(\gamma_0) C_{52}(\lfloor\gamma\rfloor - 1)$$

$$C_{59}(\gamma, \gamma_0) = 2^{1+0\vee\frac{\gamma+1-\gamma_0}{k-\gamma_0}} C_{59}(\gamma-1, \gamma_0)\lfloor\gamma\rfloor^2 C_{58}(\gamma_0) C_{52}(\lfloor\gamma\rfloor-1) C_{57}(\gamma, \gamma_0) M^\gamma \left[2 + C_{43}(\gamma', \gamma) C_{43}(\lfloor\gamma\rfloor, \gamma)\right]$$

Remark 5.13. When $\gamma \ge 1$, the definitions of $C_{59}(\gamma, \gamma_0)$ are analogous to those of $C_{59}(\gamma, 0)$ given in lemma 5.9, by replacing 0 with γ_0, by a factor $2^{1+0\vee\frac{\gamma+1-\gamma_0}{k-\gamma_0}}$.

Scheme of the proof. We give the scheme of the proof in order to explain the additional α_s in the definition of Δ_s (this additional α_s makes necessary our modification of Yoccoz's proof of proposition 5.10).

If $\gamma_0 - 1 \le \gamma < 1$, we proceed as in lemma 5.9. If $\gamma \le \gamma_0 - 1$ and $\gamma < 1$, we apply corollary 5.5 and proposition 5.7. The induction step is analogous to the proof of lemma 5.9, except the end: indeed, by proceeding as in lemma 5.9, we have:

$$A_{i,l} \le C_{59}(l, \gamma_0) C_{52}(l) C_{57}(\gamma' + r - l, \gamma_0) M^{\gamma'+r-l} q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{0\vee\frac{r-l+\gamma'+1-\gamma_0}{k-\gamma_0} + 0\vee\frac{l+1-\gamma_0}{k-\gamma_0}}$$

$$B_{i,l} \le C_{57}(r-l, 0) C_{52}(l)(r-l) M^{r-l} C_{43}(l, l+\gamma') C_{43}(\gamma', l+\gamma') C_{59}(l+\gamma', 0) q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{0\vee\frac{r-l+1-\gamma_0}{k-\gamma_0} + 0\vee\frac{l+\gamma'+1-\gamma_0}{k-\gamma_0}}$$

$$C_{i,l} \le M^{r-l} C_{52}(l) C_{57}(r-l, 0) C_{59}(l+\gamma', 0) q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{0\vee\frac{r-l+1-\gamma_0}{k-\gamma_0} + 0\vee\frac{l+\gamma'+1-\gamma_0}{k-\gamma_0}}$$

We have:

$$\left(0 \vee \frac{r-l+1-\gamma_0}{k-\gamma_0} + 0 \vee \frac{l+\gamma'+1-\gamma_0}{k-\gamma_0}\right) \vee \left(0 \vee \frac{r-l+\gamma'+1-\gamma_0}{k-\gamma_0} + 0 \vee \frac{l+1-\gamma_0}{k-\gamma_0}\right) \le 0 \vee \frac{\gamma+1-\gamma_0}{k-\gamma_0} + \frac{1}{k-\gamma_0}$$

Moreover, since $2q_{s+1}\Delta_s \ge 2q_{s+1}\alpha_s \ge 1$, then

$$A_{i,l} + B_{i,l} + C_{i,l} \le 2^{\frac{1}{k-\gamma_0} + 0\vee\frac{\gamma+1-\gamma_0}{k-\gamma_0}} C_{60}(l, \gamma', \gamma_0) q_{s+1}^{-1} q_s^{-1} (\Delta_s q_{s+1})^{0\vee\frac{\gamma+1-\gamma_0}{k-\gamma_0} + \frac{1}{k-\gamma_0}}$$

with:

$$C_{60}(l, \gamma', \gamma_0) = C_{59}(l+\gamma', \gamma_0) C_{52}(l) C_{57}(r-l+\gamma', \gamma_0) M^{r-l} \left[M^{\gamma'} + (r-l) C_{43}(\gamma', l+\gamma') C_{43}(l, l+\gamma') + 1\right]$$

(this is why we define $\Delta_s = |D^{k-1} \log Df^{q_s}|_0 + \alpha_s$. If we defined $\Delta_s = |D^{k-1} \log Df^{q_s}|_0$ and if $|D^{k-1} \log Df^{q_s}|_0$ was too small, we could not do this estimate). $\square$

By using estimation (53) and lemma 5.12, we obtain, for $0 \le n \le (q_{s+1})/q_s$, and $0 \le \gamma \le \gamma_1$ [14, p.357]:

$$\| \log Df^{nq_s} \|_\gamma \le C_{63}(\gamma, \gamma_0) q_s^{\rho(\gamma,\gamma_0)} \tag{54}$$

with

$$\rho(\gamma, \gamma_0) = \frac{(2+\beta)(0 \vee (\gamma + 1 - \gamma_0))}{k - \gamma_0} - 1$$

and

$$C_{63}(\gamma, \gamma_0) = C_{59}(\gamma, \gamma_0) \left(C_d^{-1} C_{61} \right)^{\frac{\gamma+1-\gamma_0}{k-\gamma_0} \vee 0}$$

Notice that for any

$$\gamma_1 < g(\gamma_0) = \frac{(1+\beta)\gamma_0 + k - (2+\beta)}{2+\beta} \tag{55}$$

we have $\rho(\gamma_1, \gamma_0) < 0$ (we will take $\gamma_1 = \frac{1}{2}(g(\gamma_0) + \gamma_0)$).

This implies $\sum_{s \ge 0} q_s^{\rho(\gamma_1,\gamma_0)} < +\infty$, which will allow estimating $\| \log Df^N \|_{\gamma_1}$, as we will see in the next subsection.

A remark on the method and notation: we establish estimate (54) for any $0 \le \gamma \le \gamma_1$ (and not just for γ_1) because we need it for the estimate of the quantity Z defined below.

5.3 Estimation of $\| \log Df^N \|_{\gamma_1}$

Proposition 5.14. *Let N be an integer and let us write $\gamma_1 = r + \gamma_1'$, with $0 \le \gamma_1' < 1$ and r integer. We have:*

$$\| \log Df^N \|_{\gamma_1} \le C_{63}(\gamma_1, \gamma_0) \prod_{s=1}^{\infty} \left(1 + \frac{C_{64}(\gamma_1, \gamma_0) + C_{66}(\gamma_1, \gamma_0)}{q_s^{-\rho(\gamma_1,\gamma_0)}} \right) = C_{67} \tag{56}$$

with:

$$C_{64}(\gamma_1, \gamma_0) = C_{45}(r) C_{55}(\gamma_1') C_{63}(\gamma_1, \gamma_0) C_{54}(\gamma_1')$$

$$C_{66}(\gamma_1, \gamma_0) = (r-1) C_{64}(\gamma_1, \gamma_0) C_{52}(r-1) \left(2 + C_{43}(\gamma_1', \gamma_1) C_{43}(r, \gamma_1)\right)$$

Scheme of the proof. We write $N = \sum_{s=0}^{S} b_s q_s$ with $0 \le b_s \le \frac{q_{s+1}}{q_s}$ and b_s integer. Let $N_s = \sum_{t=0}^{s} b_t q_t$ for $0 \le s \le S$. Moreover, let us write $\gamma_1 = r + \gamma_1'$, with $0 \le \gamma_1' < 1$ and r integer. By formula (78) in appendix 6.4, we can write $D^r \log Df^{N_s} = X + Y + Z$ with:

$$\begin{aligned}
X &= (D^r \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^r; \\
Y &= D^r \log Df^{N_{s-1}}; \\
Z &= \sum_{l=1}^{r-1} (D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l} G_l^r; \\
G_l^r &= G_l^r \left(D \log Df^{N_{s-1}}, ..., D^l \log Df^{N_{s-1}} \right)
\end{aligned}$$

We successively estimate X and Z. For X, we use estimate (46), corollary 5.5 and lemma 5.6 with $\phi = D^r \log Df^{b_s q_s}$ and $\psi = f^{N_{s-1}}$. We also use estimate (54), and the fact that $q_s^{\rho(r,\gamma_0)} \leq q_s^{\rho(\gamma_1,\gamma_0)}$. We get:

$$\|X\|_{\gamma_1'} \leq C_{64} q_s^{\rho(\gamma_1,\gamma_0)} (1 + \|\log Df^{N_{s-1}}\|_{\gamma_1'})$$

with

$$C_{64}(\gamma_1, \gamma_0) = C_{45}(r) C_{55}(\gamma_1') C_{63}(\gamma_1, \gamma_0) C_{54}(\gamma_1')$$

We estimate Z. By applying estimation (9), we have:

$$\|Z\|_{\gamma_1'} \leq \sum_{l=1}^{r-1} |(D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}|_0 |G_l^r|_{\gamma_1'} + \|(D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}\|_{\gamma_1'} |G_l^r|_0$$

As with X, we have:

$$\|(D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}\|_{\gamma_1'} \leq C_{65} q_s^{\rho(\gamma_1,\gamma_0)} (1 + \|\log Df^{N_{s-1}}\|_{\gamma_1'})$$

with:

$$C_{65}(\gamma_1, \gamma_0, l) = C_{45}(r-l) C_{55}(\gamma_1') C_{63}(\gamma_1' + r - l, \gamma_0) C_{54}(\gamma_1') \leq C_{64}(\gamma_1, \gamma_0)$$

Moreover, by estimate (54), we also have:

$$|(D^{r-l} \log Df^{b_s q_s} \circ f^{N_{s-1}})(Df^{N_{s-1}})^{r-l}|_0 \leq M^{r-l} C_{63}(r-l, \gamma_0) q_s^{\rho(r-l,\gamma_0)} \leq C_{64} q_s^{\rho(\gamma_1,\gamma_0)}$$

For G_l^r, we use lemma 5.4 with the polynomial $P = G_l^r$ (see appendix 6.4). We estimate $\|Z\|_{\gamma_1'}$ by applying estimation (42) twice. We get:

$$\|Z\|_{\gamma_1'} \leq C_{66} q_s^{\rho(\gamma_1,\gamma_0)} \|\log Df^{N_{s-1}}\|_{\gamma_1}$$

with:

$$C_{66}(\gamma_1, \gamma_0) = (r-1) C_{64}(\gamma_1, \gamma_0) C_{52}(r-1) \left(2 + C_{43}(\gamma_1', \gamma_1) C_{43}(r, \gamma_1)\right)$$

Therefore, since $\|Y\|_{\gamma_1'} = \|D^r \log Df^{N_{s-1}}\|_{\gamma_1'}$, we get, for $s \geq 1$:

$$\|D^r \log Df^{N_s}\|_{\gamma_1'} \leq \|\log Df^{N_s}\|_{\gamma_1} \leq \left(1 + \frac{C_{64}(\gamma_1, \gamma_0) + C_{66}(\gamma_1, \gamma_0)}{q_s^{-\rho(\gamma_1,\gamma_0)}}\right) \|D^r \log Df^{N_{s-1}}\|_{\gamma_1'}$$

Moreover, by estimate (54), since $N_0 = b_0$, we also have:

$$\|D^r \log Df^{N_0}\|_{\gamma_1'} \leq C_{63}(\gamma_1, \gamma_0)$$

We conclude that:

$$\|\log Df^N\|_{\gamma_1} \leq C_{63}(\gamma_1, \gamma_0) \prod_{s=1}^{\infty} \left(1 + \frac{C_{64}(\gamma_1, \gamma_0) + C_{66}(\gamma_1, \gamma_0)}{q_s^{-\rho(\gamma_1,\gamma_0)}}\right) = C_{67}$$

□

5.3.1 Computation of the estimations: proof of proposition 5.1.

The quantity C_{67} depends on $\sup_{p\geq 0} \|\log Df^p\|_{\gamma_0}$. We estimate C_{67}. First, we estimate C_{61}. Since $5C_{62} \leq (C_{59}(k-1,0))^2$, we estimate $C_{59}(\gamma,0)$ for $0 \leq \gamma \leq k-1$. By combining the constants appearing in lemma 5.9, we get:

$$C_{59}(\gamma,0) \leq C_{59}(\gamma',0)\left[(k-1)2C_{58}(0)C_{57}(k-1,0)C_{52}(k-1)M^k(\log MC_{42}(k))^2\right]^{k-1}$$

Moreover,

$$C_{52}(k-1) \leq (4k)^{4k}2^{k-1}\left(C_{42}(k)\log M\right)^{k-1}$$

$$C_{53}(k-1) \leq (4k)^{4k}kM^{k-1}2^{k-1}\left(C_{42}(k)(M+1)\right)^{k-1}\left(1+((M+1)C_{42}(k))^2\right)$$

$$C_{53}(k-1) \leq (4k)^{4k}k2^{2k+1}M^{2k}\left(C_{42}(k)\right)^{k+1}$$

Note that for any $l \geq 2$, $C_{54}(l) = C_{53}(l)$. Since

$$C_{57}(k-1,0) \leq [C_{54}(k-1)]^2 2MC_{42}(k)$$

We get:

$$C_{57}(k-1,0) \leq (4k)^{8k}k^2 2^{4k+3}(C_{42}(k))^{2k+3}M^{4k+1} = \tilde{C}_{57}(k-1,0) \tag{57}$$

We have:

$$C_{59}(\gamma,0) \leq C_{59}(\gamma',0)\left[C_{58}(0)(\tilde{C}_{57}(k-1,0))^2M^k\right]^{k-1}$$

Since $C_{58}(0) \leq (2M^{\frac{3}{2}(k-1)}C_{27}^{f,k}C_d^{-1})^{1/k}$, we get, for $0 \leq \gamma \leq k-1$:

$$C_{59}(\gamma,0) \leq C_d^{-1}C_{27}^{f,k}2M^{k^2+\frac{3}{2}(k-1)}\left[\tilde{C}_{57}(k-1,0)\right]^{2k} = \tilde{C}_{59} \tag{58}$$

We get:

$$C_d^{-1}C_{61} \leq C_d^{-1}C_{27}^{f,k}M^{k-1}\prod_{s=0}^{\infty}\left(1+\frac{(\tilde{C}_{59})^2}{q_s^{\epsilon}}\right) \leq \prod_{s=0}^{\infty}\left(1+\frac{2C_d^{-1}C_{27}^{f,k}M^{k-1}(\tilde{C}_{59})^2}{q_s^{\epsilon}}\right) \tag{59}$$

We estimate $C_{64}(\gamma_1,\gamma_0)$ and $C_{66}(\gamma_1,\gamma_0)$.
We have: $C_{64}(\gamma_1,\gamma_0) \leq 2(r+1)M^{r+1}C_{63}(\gamma_1,\gamma_0)C_{55}(\gamma_1')$.
Therefore,

$$C_{64}(\gamma_1,\gamma_0) \leq (\max_{0\leq\gamma\leq k-1} C_{59}(\gamma,\gamma_0))(C_d^{-1}C_{61})^{\frac{k-1}{k-\gamma_0}}C_{55}(\gamma_1')2(r+1)M^{r+1}$$

We estimate $C_{66}(\gamma_1,\gamma_0)$. We have:

$$C_{66}(\gamma_1,\gamma_0) \leq (r-1)(2+((\log M)C_{42}(k)))^2C_{64}(\gamma_1,\gamma_0)C_{52}(r-1) \tag{60}$$

To complete the estimations of $C_{64}(\gamma_1,\gamma_0)$ and $C_{66}(\gamma_1,\gamma_0)$, we need to estimate $C_{59}(\gamma,\gamma_0)$. By writing $\gamma = \lfloor\gamma\rfloor+\gamma'$, and by proceeding as for the estimation of $C_{59}(\gamma,0)$, we have:

$$C_{59}(\gamma,\gamma_0) \le C_{59}(\gamma',\gamma_0)(C_d^{-1}C_{61})^{\frac{k-1}{k-\gamma_0}} C_{57}(k-1,\gamma_0)^{2k} M^{k^2} 2^{2(k-1)}$$

Moreover,

$$C_{57}(k-1,\gamma_0) \le C_{57}(k-1,0)C_{55}(k-1)$$

We can also check that:

$$C_{59}(\gamma',\gamma_0) = 2C_{42}(k)M^{3\gamma'+1} \vee 2M^{\gamma'+1}C_{55}(\gamma'+1) \le \tilde{C}_{57}(k-1,0)C_{55}(k-1)$$

Therefore,

$$C_{64}(\gamma_1,\gamma_0) \le (C_d^{-1}C_{61})^{\frac{2(k-1)}{k-\gamma_0}} \left(\tilde{C}_{57}(k-1,0)\right)^{2k+1} (C_{55}(k-1))^{2k+2}\, 2kM^{k^2+k} \tag{61}$$

Now, let

$$\tau_1 = \frac{5(k-1)}{\beta+2+\eta} \tag{62}$$

let $C_{68} = (k+2)^{400k^4} M^{24k(k+1)} (C_{27}^{f,k})^{4k} (C_d^{-1})^{4k^2}$. Let also

$$\epsilon_1 = \min\left(\epsilon, \frac{\eta}{2(\beta+2+\eta)}\right) \tag{63}$$

We have: $\epsilon_1 \le \min(\epsilon, -\rho(\gamma_1,\gamma_0))$ and for any $\gamma_0 \le k-2-\beta-\eta$, we have $\tau_1 \ge \frac{2(k-1)}{k-\gamma_0}$.

Note that $C_{42}(k)$ only depends on k, and that $\tilde{C}_{57}(k-1,0)$ and $C_{55}(k-1)$ only depend on k and M.

By combining estimations (57), (58), (59), (61) and (60), we can check that we have:

$$C_{67} \le \prod_{s=0}^{\infty} \left(1 + \frac{C_{68} \prod_{s=0}^{\infty} \left(1 + \frac{C_{68}}{q_s^{\epsilon_1}}\right)^{\tau_1}}{q_s^{\epsilon_1}}\right)$$

Since $q_s \ge (\sqrt{2})^{s-1}$, we get:

$$C_{67} \le \prod_{s\ge 0} \left(1 + \frac{\sqrt{2}C_{68} \prod_{s\ge 0} \left(1 + \frac{\sqrt{2}C_{68}}{2^{s\frac{\epsilon_1}{2}}}\right)^{\tau_1}}{2^{s\frac{\epsilon_1}{2}}}\right)$$

In order to obtain the final estimation, we need the claim:

Claim 5.15. *Let $C \ge 10$. For any $2 \ge u > 1$, we have:*

$$\prod_{n=0}^{\infty}\left(1 + \frac{C}{u^n}\right) \le e^{\frac{2^{2/3}}{\log u}(\log C)^2}$$

Proof.

$$\begin{aligned}\sum_{n\geq 0}\log(1+C/u^n) &= \sum_{n\leq \frac{\log C}{\log u}-1}\log(1+C/u^n)+\sum_{n>\frac{\log C}{\log u}-1}\log(1+C/u^n)\\ &\leq \frac{\log C}{\log u}\log(1+C)+\sum_{n>\frac{\log C}{\log u}-1} C/u^n\\ &\leq \frac{\log C}{\log u}(\log(1+C)+1)\leq \frac{2^{2/3}}{\log u}(\log C)^2\end{aligned}$$

for $C \geq 10$.

□

By applying this proposition twice, we get the claim:

Claim 5.16. *Let $C \geq 10$, $2 \geq u > 1$, $\tau \geq 1$. We have:*

$$\prod_{n\geq 0}\left(1+\frac{C\prod_{n=0}^{\infty}\left(1+\frac{C}{u^n}\right)^{\tau}}{u^n}\right)\leq e\wedge\left(\frac{18\tau^2}{(\log u)^3}(\log C)^4\right)$$

Let $C_{69} = \sqrt{2}C_{68}$. We apply claim 5.16 with $C = C_{69}$, $u = 2^{\frac{\epsilon_1}{2}}$, $\tau = \tau_1$. We obtain:

$$C_{67} \leq e\wedge\left(\frac{18\tau_1^2}{(\frac{\epsilon_1}{2}\log 2)^3}(\log C_{69})^4\right) \tag{64}$$

Moreover, let

$$C_{70} = \frac{18\tau_1^2}{(\frac{\epsilon_1}{2}\log 2)^3} \tag{65}$$

and let $C_{71}^{-} = \frac{18\times 5^2\times 8}{(\log 2)^3}$ a numerical constant. We have:

$$C_{70} \leq C_{71}^{-}\frac{(k-1)^2}{\epsilon_1^2}$$

By using the definitions of ϵ_1 (see (63)) and τ_1 (see (62)), since $\epsilon = \frac{k-(1+2\beta)}{2k}$ and since $\eta \leq k-2-\beta$, we have:

$$C_{70} \leq C_{71}^{-}\frac{k^2}{\left(\min\left(\frac{k-(2\beta+1)}{2k},\frac{\eta}{2(\beta+2+\eta)}\right)\right)^3} \leq 8C_{71}^{-}\frac{k^5}{(\min(k-(2\beta+1),k-(\beta+2)))^3} = C_{72}(k,\beta)$$

Therefore, we get:

$$\|\log Df^N\|_{\gamma_1} \leq e\wedge\left(C_{72}(k,\beta)\left(C_{73}^{f,k}+4k^2\log(C_d^{-1})+24k(k+1)\sup_{p\geq 0}\|\log Df^p\|_{\gamma_0}\right)^4\right) \tag{66}$$

with:

$$C_{73}^{f,k} = \log\left(\sqrt{2}(k+2)^{400k^4}(C_{27}^{f,k})^{4k}\right)$$

Hence proposition 5.1.

5.3.2 Proof of theorem 1.6: estimations (6) and (7).

By corollary 5.5, we have:

$$\|Df^N\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq C_{54}(k-1)(1+\|\log Df^N\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}})$$

Moreover, we recall that:

$$C_{54}(k-1) \leq (4k)^{4k} k 2^{2k+1} M^{2k} \left(C_{42}(k)\right)^{k+1}$$

We have: $C_{54}(k-1) \leq C_{74}(k) M^{2k}$ with $C_{74}(k) = (4k)^{4k} k 2^{(k+1)^3+3k+2}$

Moreover, $\|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq \sup_{N\geq 0} \|Df^N\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}}$. We get:

$$\|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq e\wedge\Big(C_{72}(k,\beta)\left(C_{73}^{f,k} + 4k^2\log C_d^{-1} + 24k(k+1)\log M'\right)^4 + \log(C_{74}(k)) + 2k\log M'\Big)$$

Since $2k\log M' \leq k(k+1)\log M'$, we get:

$$\|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq e\wedge\Big(C_{72}(k,\beta)\left(C_{73}^{f,k} + 4k^2\log C_d^{-1} + 25k(k+1)\log M' + \log(C_{74}(k))\right)^4\Big) \tag{67}$$

We show estimation (6). We suppose $k \geq 3\beta + 9/2$. Let:

$$C_{75}(\beta, C_d, |Df|_0, W(f), |Sf|_0, |Sf|_{k-3}) = e^{(3)}\wedge(C_3(\beta)C_4(C_d)C_5(|Df|_0, W(f), |Sf|_0)C_6(|Sf|_{k-3}))$$

i.e. we consider the bound given by estimation (2), except that we replace $|Sf|_{\lceil 3\beta+3/2\rceil}$ with $|Sf|_{k-3}$.

$C_{27}^{f,k}$ depends on k, $|Sf|_{k-3}$ and $W(f)$. We have:

$$4k\log C_{27}(k, |Sf|_{k-3}, W(f)) \leq 4k(k+4)!\log C_{27}(3\beta+9/2, |Sf|_{k-3}, W(f)) \leq 4(k+5)!\log C_{75}$$

Moreover, since $M' \leq C_{75}$ and $k \geq 5$,

$$4k^2\log C_d^{-1} + 25k(k+1)\log M' + \log(C_{74}(k)) + 400k^4\log(\sqrt{2}(k+2)) \leq C_{76}(k)\log C_{75}$$

with $C_{76} = 4k^2 + 25k(k+1) + \log(C_{74}(k)) + 400k^4\log(\sqrt{2}(k+2)) \leq (k+5)!$. Therefore,

$$\|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq e\wedge\left(C_{72}(k,\beta)\left(5(k+5)!\log C_{75}\right)^4\right)$$

We also have:

$$C_{72}(k,\beta) \leq 8C_{71}^{-}\frac{k^5}{(\min(k-(2\beta+1), k-(\beta+2)))^3} \leq C_{71}^{-}k^5$$

Since $C_{71}^{-}k^5(5(k+5)!)^4 \leq C_{71}^{-}5^4((k+7)!)^4$, and since $2 \geq 2\log 2$, we conclude:

$$\|Dh\|_{\frac{k}{2(\beta+2)}-\frac{1}{2}} \leq e\wedge\left(C_{12}(k)e^{(2)}\wedge(2 + C_3(\beta)C_4(C_d)C_5(|Df|_0, W(f), |Sf|_0)C_6(|Sf|_{k-3}))\right)$$

with:

$$C_{12}(k) = \frac{18\times 5^6\times 8}{(\log 2)^3}((k+7)!)^4$$

If $\beta = 0$, we can use the C^1 estimate. We have: $\log M' \leq C_1/C_d$ and therefore,

$$4k^2 \log(C_d^{-1}) + 25k(k+1)\log M' \leq (k+5)! C_1/C_d$$

Moreover,

$$C_{73}^{f,k} + \log(C_{74}(k)) \leq 4(k+5)! \left(\log(1 \vee |Sf|_{k-3}) + W(f) + 1\right)$$

then by using estimation (67), we obtain:

$$\|Dh\|_{\frac{k}{4}-\frac{1}{2}} \leq e \wedge \left(C_{13}(k)\left[C_{14}[W(f), |Sf|_{k-3}] + \frac{C_1[W(f), |Sf|_0]}{C_d}\right]^4\right)$$

with:

$$C_{13}(k) = C_{12}(k)$$

$$C_{14}[W(f), |Sf|_{k-3}] = \log(1 \vee |Sf|_{k-3}) + W(f) + 1$$

5.4 Iteration of the reasoning: proof of estimation (5) of theorem 1.6.

To obtain an estimation of the $C^{k-1-\beta-\eta}$-norm of the conjugacy, we iterate estimation (41). We take $\gamma_0 = \gamma_i$ and $\gamma_1 = \gamma_{i+1} = \frac{1}{2}(g(\gamma_i) + \gamma_i)$. Thus, $\gamma_{i+1} < g(\gamma_i)$ and $\lim_{i\to+\infty} \gamma_i = k-2-\beta$. We need to estimate the rank above which $\gamma_i \geq k-2-\beta-\eta$:

Claim 5.17. *Let $C_{77} = \frac{\beta+3/2}{\beta+2}$, $C_{78} = \frac{k-2-\beta}{2(\beta+2)}$. If*

$$n \geq \log\left(\frac{C_{78}}{\eta(1-C_{77})}\right) / \log\left(\frac{1}{C_{77}}\right) = C_{79}$$

we have $\gamma_n \geq k-2-\beta-\eta$.

Proof. We have: $\gamma_{n+1} = C_{77}\gamma_n + C_{78}$. Therefore, $\gamma_n = C_{78}\frac{1-C_{77}^n}{1-C_{77}}$. Therefore, $|\gamma_n - (k-2-\beta)| = \left|C_{78}\frac{C_{77}^n}{1-C_{77}}\right| \leq \eta$ if $n \geq C_{79}$. □

Claim 5.18. *Let $F(x) = e^{c(a+bx)^4}$. For any $x, c \geq 1$, $a, b \geq 4$, and integer $n \geq 1$, we have:*

$$F^n(x) \leq e^{(n)} \wedge \left((3+n)c(a+bx)^4\right)$$

Proof. We show this estimate by induction. If $n = 1$, this estimate holds. Suppose this estimate holds at rank n. We have:

$$F^{n+1}(x) = F^n\left(e^{c(a+bx)^4}\right) \leq e^{(n)} \wedge \left((3+n)c(a+be^{c(a+bx)^4})^4\right)$$

For any $x \geq 4$, $e^x \geq x^2$. Since $c(a+bx)^4 \geq 4$ and $4n \geq 3+n$, then:

$$e^{nc(a+bx)^4} \geq \left(nc(a+bx)^4\right)^2 \geq (3+n)c(a+b)^4$$

$$e^{(4+n)c(a+bx)^4} \geq (3+n)c(a+b)^4 e^{4c(a+bx)^4} \geq (3+n)c(a+be^{c(a+bx)^4})^4$$

Hence the estimate at rank $n+1$. □

We apply proposition 5.1. In claim 5.18, we take $x = \log M'$, $a = C_{73}^{f,k} + 4k^2 \log(C_d^{-1})$, $b = 24k(k+1)$, $c = C_{72}(k,\beta)$. Let

$$n_7 = \left\lceil \frac{\log((k-2-\beta)/\eta)}{\log(1+1/(2\beta+3))} \right\rceil$$

We have: $C_{79} + 1 > n_7 \geq C_{79}$. We get:

$$\|\log Df^N\|_{k-2-\beta-\eta} \leq e^{(n_7)} \wedge \Big((3+n_7)C_{72}(k,\beta)(C_{73}^{f,k} + 4k^2\log(C_d^{-1}) + 24k(k+1)\log M')^4\Big)$$

Moreover, by corollary 5.5, we have:

$$\|Df^N\|_{k-2-\beta-\eta} \leq C_{54}(1 + \|\log Df^N\|_{k-2-\beta-\eta})$$

Since $\|Dh\|_{k-2-\beta-\eta} \leq \|Df^N\|_{k-2-\beta-\eta}$, we get:

$$\|Dh\|_{k-2-\beta-\eta} \leq e^{(n_7)} \wedge \left((4+n_7)C_{72}(k,\beta)\left[C_{73}^{f,k} + 4k^2\log(C_d^{-1}) + 25k(k+1)\log M'\right]^4\right)$$

Since $M' \leq C_2$, we let:

$$C_{11}[\eta,k,\beta,C_d,|Df|_0,W(f),|Sf|_{k-3}] = (4+n_7)C_{72}(k,\beta)\left[C_{73}^{f,k} + 4k^2\log(C_d^{-1}) + 25k(k+1)\log C_2\right]^4$$

We recall that:

$$n_7 = \left\lceil \frac{\log((k-2-\beta)/\eta)}{\log(1+1/(2\beta+3))} \right\rceil$$

$$C_{72}(k,\beta) = \frac{18 \times 5^2 \times 8^2}{(\log 2)^3} \frac{k^5}{(\min(k-(2\beta+1), k-(\beta+2)))^3}$$

$$C_{73}^{f,k} \leq (k+7)!\,(1 + W(f) + \log(1 \vee |Sf|_{k-3}))$$

We have:

$$\|Dh\|_{k-2-\beta-\eta} \leq e^{(\lceil \log((k-2-\beta)/\eta)/\log(1+1/(2\beta+3))\rceil)} \wedge (C_{11}[\eta,k,\beta,C_d,|Df|_0,W(f),|Sf|_{k-3}]) \tag{68}$$

6 Appendix: Omitted Proofs

6.1 Proof of lemma 4.4

We follow [14] but we give more details. Let $p \leq q_{n+1}$. The case $r = 1$ stems from lemma 4.3. For the case $r = 2$, we also use lemma 4.3:

$$|D^2 \log Df^p(x)| \leq |Sf^p(x)| + \frac{1}{2}|D\log Df^p(x)|^2 \leq \left(C_{24}^f + \frac{1}{2}(C_{26}^f)^2\right)\frac{M_n}{m_n(x)^2}$$

In particular, we can take

$$C^f_{27}(2) = 82|Sf|_0 e^{8W(f)}$$

For $r > 2$, we prove lemma 4.4 by induction. Suppose the lemma is proved up to $r \geq 2$. Since for any C^3-diffeomorphisms g and h,

$$S(g \circ h) = (Sg \circ h)(Dh)^2 + Sh$$

then for $p \geq 1$,

$$Sf^p = \sum_{i=0}^{p-1} (Sf \circ f^i)(Df^i)^2$$

and by differentiating this last equality, we get, for $r \geq 0$, $n \geq 1$,

$$D^r Sf^p = \sum_{l=0}^{r} \sum_{i=0}^{p-1} (D^{r-l} Sf \circ f^i)(Df^i)^{r-l+2} F^r_l(D \log Df^i, ..., D^l \log Df^i) \tag{69}$$

where F^r_l is a polynomial in l variables $X_1, ..., X_l$, homogenous of weight l if X_i is given the weight i. Moreover, since $Sf = D^2 \log Df - \frac{1}{2}(D \log Df)^2$, then for $r \geq 2$,

$$D^{r-2} Sf = D^r \log Df + G_r(D \log Df, ..., D^{r-1} \log Df) \tag{70}$$

where G_r is a polynomial in $r - 1$ variables $X_1, ..., X_{r-1}$, homogeneous of weight r if X_i is given the weight i. Therefore, in order to estimate $|D^r \log Df|_0$, it suffices to estimate $F^r_l(D \log Df^i, ..., D^l \log Df^i)$ and $G_r(D \log Df, ..., D^{r-1} \log Df)$. These estimations are given by lemmas 6.1 and 6.2. They are used in [14] but we recall them here in order to compute the constants $C^f_{80}(r)$ in lemma 6.1 and $C^f_{81}(r)$ in lemma 6.2.

Lemma 6.1. *Under the induction assumption, for* $0 \leq l \leq r$ *and* $0 \leq p \leq q_{n+1}$*, we have:*

$$|F^r_l(D \log Df^p(x), ..., D^l \log Df^p(x))| \leq C^f_{80}(r) \left[\frac{M_n^{1/2}}{m_n(x)} \right]^l$$

with:

$$C^f_{80}(r) = (r)! \frac{(2r)!}{2} \left(C^f_{27}(r) \right)^r$$

Proof. We follow [14]. By derivating equation (69), we get:

$$D^{r+1} Sf^p = \sum_{i=0}^{n-1} \sum_{l=0}^{r} (D^{r+1-l} Sf \circ f^i)(Df^i)^{r+1-l+2} F^r_l(D \log Df^i, ..., D^l \log Df^i) +$$

$$(D^{r-l} Sf \circ f^i)(Df^i)^{r-l+2}(r - l + 2) D \log Df^i$$

$$F^r_l(D \log Df^i, ..., D^l \log Df^i) + \sum_{j=1}^{l} \frac{\partial F^r_l}{\partial X_j}(D \log Df^i, ..., D^l \log Df^i) D^{j+1} \log Df^i (Df^i)^{r-l+2}$$

$$D^{r+1}Sf^p = \sum_{i=0}^{p-1}\sum_{l=0}^{r}(D^{r+1-l}Sf\circ f^i)(Df^i)^{r+1-l+2}F_l^r(D\log Df^i,...,D^l\log Df^i) +$$

$$\sum_{l=1}^{r+1}(D^{r+1-l}Sf\circ f^i)(Df^i)^{r-l+3}(r-l+3)D\log Df^i F_{l-1}^r(D\log Df^i,...,D^{l-1}\log Df^i) +$$

$$\sum_{l=1}^{r+1}\sum_{j=2}^{l}\frac{\partial F_{l-1}^r}{\partial X_{j-1}}(D\log Df^i,...,D^{l-1}\log Df^i)D^j\log Df^i(Df^i)^{r+1-l+2}$$

Therefore, for $1\leq l\leq r$,

$$F_l^{r+1} = F_l^r + (r-l+3)X_1F_{l-1}^r + \sum_{j=2}^{l}X_j\frac{\partial F_{l-1}^r}{\partial X_{j-1}} \tag{71}$$

for $l=0$,

$$F_l^{r+1} = F_l^r$$

and for $l=r+1$,

$$F_l^{r+1} = (r-l+3)X_1F_{l-1}^r + \sum_{j=2}^{l}X_j\frac{\partial F_{l-1}^r}{\partial X_{j-1}}$$

Now, let us write

$$F_l^r = \sum_{i_1+2i_2+...+li_l=l} a_{l,r}(i_1,...,i_l)X_1^{i_1}...X_l^{i_l}$$

We have $a_{l,r}(i_1,...,i_l)\geq 0$. Let

$$a_{l,r} = \max_{i_1+2i_2+...+li_l=l} a_{l,r}(i_1,...,i_l)$$

and

$$\bar{a}_r = \max_{0\leq j\leq r} a_{j,r}$$

Consider $i_1,...,i_l$ such that $a_{l,r}(i_1,...,i_l)=a_{l,r}$. By applying equation (71), we have, for $1\leq l\leq r$:

$$a_{l,r+1}\leq a_{l,r}+(r+3-l)a_{l-1,r}+(l-1)(\max i_j)a_{l-1,r}\leq (r+3-l+l^2-l)\bar{a}_{l,r}\leq (r+1)^2\bar{a}_{l,r}$$

For $l=0$ or $r+1$, this estimate still holds. Therefore, $\bar{a}_{r+1}\leq (r+1)^2\bar{a}_r$ and by iteration, we obtain:

$$\bar{a}_r\leq (r!)^2$$

Moreover, since

$$F_l^r(D\log Df^i,...,D^l\log Df^i) = \sum_{i_1+2i_2+...+li_l=l} a_{l,r}(i_1,...,i_l)(D\log Df^i)^{i_1}...(D^l\log Df^i)^{i_l}$$

and since $\#\{(i_1,...,i_l)/i_1+2i_2+...+li_l=l\} \leq \#\{(i_1,...,i_l)/i_1+i_2+...+i_l=l\} = \frac{(2l-1)!}{l!(l-1)!}$ (this classical equality can be shown by induction) then by applying the induction assumption,

$$|F_l^r(D\log Df^i(x),...,D^l\log Df^i(x))| \leq (r!)^2\frac{(2l-1)!}{l!(l-1)!}\max_{i_1+2i_2+...+li_l=l}(C_{27}^f(1))^{i_1}...(C_{27}^f(l))^{li_l}\left[\frac{M_n^{1/2}}{m_n(x)}\right]^l$$

and since the $C_{27}^f(i)$ are increasing with i, we obtain:

$$\left|F_l^r(D\log Df^i(x),...,D^l\log Df^i(x))\right| \leq C_{80}^f(r)\left[\frac{M_n^{1/2}}{m_n(x)}\right]^l$$

□

Likewise, the estimation of $G_r(D\log Df^p,...,D^{l-1}\log Df^p)$ is given by the lemma:

Lemma 6.2. *For any $x\in\mathbb{T}^1$, $0\leq p\leq q_{n+1}$, $r\geq 2$,*

$$|G_r(D\log Df^p(x),...,D^{l-1}\log Df^p(x))| \leq C_{81}^f(r)\left[\frac{M_n^{1/2}}{m_n(x)}\right]^r$$

with:
$C_{81}^f(r+1) = \frac{(2r)!}{2(r+1)}(C_{27}^f(r))^{r+1}$

Proof. The polynomial G_r satisfies the following identity:

$$G_{r+1} = \sum_{j=2}^{r} X_j\frac{\partial G_r}{\partial X_{j-1}}$$

We denote

$$G_r = \sum_{i_1+2i_2+...+(r-1)i_{r-1}=r} b_r(i_1,...,i_{r-1})X_1^{i_1}...X_{r-1}^{i_{r-1}}$$

(we have, for example, $G_2 = -\frac{1}{2}X_1^2$)
Let

$$b_r = \max_{i_1+2i_2+...+(r-1)i_{r-1}=r}|b_r(i_1,...,i_{r-1})|$$

For $r\geq 2$, we have $b_{r+1}\leq r(\max_{1\leq j\leq r-1} i_j)b_r \leq r^2 b_r$ and therefore, $b_r\leq\frac{(r-1)!^2}{2}$
Therefore,

$$|G_{r+1}(D\log Df^p(x),...,D^r\log Df^p(x))| \leq \frac{r!}{2}\frac{(2r)!}{r!(r+1)!}\max_{i_1+2i_2+...+ri_r=r+1}(C_{27}^f(1))^{i_1}...(C_{27}^f(r))^{i_r}\left[\frac{M_n^{1/2}}{m_n(x)}\right]^{r+1}$$

Since the constants $C_{27}^f(r)$ are increasing with r, we can take:

$$C_{81}^f(r+1) = \frac{(2r)!}{2(r+1)}(C_{27}^f(r))^{r+1}$$

□

We can now show estimation (14). By applying equation (70), we have, for $r \geq 2$:

$$D^{r+1} \log Df^p = D^{r-1} Sf^p - G_{r+1}(D \log Df^p, ..., D^r \log Df^p)$$

Therefore, by equation (69) and lemma 4.1,

$$|D^{r+1} \log Df^p(x)| \leq \left(rC_{80}^f(r)|Sf|_{r-1}e^{(r+1)W(f)} + C_{81}^f(r+1)\right)\left(\frac{M_n^{1/2}}{m_n(x)}\right)^{r+1}$$

$$|D^{r+1} \log Df^p(x)| \leq (C_{27}^f(r))^r \frac{(2r)!}{2}\left(|Sf|_{r-1}e^{(r+1)W(f)} + C_{27}^f(r)\right)\left(\frac{M_n^{1/2}}{m_n(x)}\right)^{r+1}$$

We can show by induction on r that we can take, for $r \geq 3$,

$$C_{27}^f(r) = \left[C_{27}^f(2)(2r)^{2r}(\max(1, |Sf|_{r-2}))e^{rW(f)}\right]^{r!}$$

6.2 Proof of proposition 5.3

This estimation is well-known (see e.g. the appendix in [6]). We recall the proof in order to determine the constant. We write $\gamma_2 = r_2 + \gamma_2'$, $\gamma_3 = r_3 + \gamma_3'$ with $0 < \gamma_2', \gamma_3' < 1$, and we estimate $|D^{r_2}\phi|_{\gamma_2'}$ in function of $|D^{r_3}\phi|_{\gamma_3'}$ and $|\phi|_0$. We use [5, p. 110]:

Proposition 6.3. *Let $0 < \beta < 1$ and $\phi \in C^1(\mathbb{T}^1)$. We have:*

$$|\phi|_\beta \leq 2|\phi|_0^{1-\beta}|D\phi|_0^\beta$$

Proposition 6.4. *Let $0 < \beta < 1$ and $\phi \in C^\beta(\mathbb{T}^1)$. If $\beta' < \beta$, we have:*

$$|\phi|_{\beta'} \leq 2|\phi|_0^{1-\frac{\beta'}{\beta}}|\phi|_\beta^{\frac{\beta'}{\beta}}$$

Proposition 6.5. *Let $0 < \beta \leq 1$ and $\phi \in C^1(\mathbb{T}^1)$ such that $D\phi \in C^\beta(\mathbb{T}^1)$. We have:*

$$|D\phi|_0 \leq \frac{3}{2}2^{\frac{\beta}{1+\beta}}\left(|\phi|_0^\beta|D\phi|_\beta\right)^{\frac{1}{1+\beta}}$$

Moreover, for $\phi \in C^2(\mathbb{T}^1)$, we have:

$$|D\phi|_0 \leq \left(2|\phi|_0|D^2\phi|_0\right)^{1/2}$$

First, we suppose $\gamma_2' = \gamma_3' = 0$. We already know that for any $j < l$, there exists $C_{82}(j,l)$ such that:

$$|D^j\phi|_0 \leq C_{82}(j,l)|\phi|_0^{1-\frac{j}{l}}|D^l\phi|_0^{\frac{j}{l}}$$

We want to estimate a possible $C_{82}(j,l)$ in function of l. First, we show that $C_{82}(1,l) = C_{82}(l-1,l) = (\sqrt{2})^{l-1}$. We proceed inductively on l. For $l = 2$, by proposition 6.5, $C_{82}(1,l) = C_{82}(l-1,l) = \sqrt{2}$. For $l \geq 3$, we have:

$$|D\phi|_0 \leq C_{82}(1,l-1)|\phi|_0^{\frac{l-2}{l-1}}|D^{l-1}\phi|_0^{\frac{1}{l-1}} \tag{72}$$

$$|D^{l-2}\phi|_0 \leq C_{82}(l-2,l-1)|\phi|_0^{\frac{1}{l-1}}|D^{l-1}\phi|_0^{\frac{l-2}{l-1}} \tag{73}$$

By applying (73) to $D\phi$, we also have:

$$|D^{l-1}\phi|_0 \leq C_{82}(l-2, l-1)|D\phi|_0^{\frac{1}{l-1}} |D^l\phi|_0^{\frac{l-2}{l-1}}$$

By combining this estimate with (72), we get:

$$|D^{l-1}\phi|_0^{1-\frac{1}{(l-1)^2}} \leq C_{82}(l-2, l-1)(C_{82}(1, l-1))^{\frac{1}{l-1}} |\phi|_0^{\frac{l-2}{(l-1)^2}} |D^l\phi|_0^{\frac{l-2}{l-1}}$$

Therefore,

$$C_{82}(l-1, l) = \left(C_{82}(l-2, l-1)(C_{82}(1, l-1))^{\frac{1}{l-1}}\right)^{\frac{1}{1-\frac{1}{(l-1)^2}}}$$

Likewise, we can show that

$$C_{82}(1, l) = \left(C_{82}(1, l-1)(C_{82}(l-2, l-1))^{\frac{1}{l-1}}\right)^{\frac{1}{1-\frac{1}{(l-1)^2}}}$$

Let $u_l = \log C_{82}(1, l)$, $v_l = \log C_{82}(l-1, l)$, $a_l = \frac{1}{1-\frac{1}{(l-1)^2}}$. We have:

$$u_l = a_l(u_{l-1} + \frac{1}{l-1} v_{l-1})$$

$$v_l = a_l(v_{l-1} + \frac{1}{l-1} u_{l-1})$$

Let $w_l = u_l + v_l$. We have: $w_l = \frac{l-1}{l-2} w_{l-1}$ and therefore, $w_l = (l-1)w_2$. Moreover, let $x_l = u_l - v_l$. We have: $x_l = \frac{l}{l-1} x_{l-1}$. Since $x_2 = 0$, then $x_l = 0$ for any $l \geq 3$. Therefore, $u_l = v_l = (l-1)\log\sqrt{2}$. Therefore, $C_{82}(1, l) = C_{82}(l-1, l) = (\sqrt{2})^{l-1}$.

To show that $C_{82}(j, l) \leq (\sqrt{2})^{l^2}$ for $l-2 \geq j \geq 2$, we proceed inductively on l. We know that $C_{82}(1, 2) = \sqrt{2}$ and we suppose that $C_{82}(j, l-1) \leq (\sqrt{2})^{(l-1)^2}$ for $j = 2, ..., l-2$. We observe that:

$$|D^{j+1}\phi|_0 \leq C_{82}(j, l-1)|D\phi|_0^{1-\frac{j}{l-1}} |D^l\phi|_0^{\frac{j}{l-1}}$$

Therefore,

$$C_{82}(j+1, l) \leq C_{82}(j, l-1)(C_{82}(1, l))^{\frac{l-j-1}{l-1}} = C_{82}(j, l-1)(\sqrt{2})^{l-j-1}$$

By applying the induction assumption, we get:

$$C_{82}(j+1, l) \leq (\sqrt{2})^{(l-1)^2+l-1} \leq (\sqrt{2})^{l^2}$$

Hence the proposition for γ_2, γ_3 integers.

Now, we suppose that γ_2 and γ_3 are not integers.
If $r_2 < r_3$, we have:

$$|D^{r_2}\phi|_{\gamma_2'} \leq 2|D^{r_2}\phi|_0^{1-\gamma_2'} |D^{r_2+1}\phi|_0^{\gamma_2'}$$

Moreover, by the interpolation inequality in the integer case, we have:

$$|D^{r_2+1}\phi|_0 \le C_{82}(r_3)|\phi|_0^{1-\frac{r_2+1}{r_3}}|D^{r_3}\phi|_0^{\frac{r_2+1}{r_3}}$$

$$|D^{r_2}\phi|_0 \le C_{82}(r_3)|\phi|_0^{1-\frac{r_2}{r_3}}|D^{r_3}\phi|_0^{\frac{r_2}{r_3}}$$

Therefore, by proposition 6.3

$$|D^{r_2}\phi|_{\gamma_2'} \le 2C_{82}(r_3)|\phi|_0^{(1-\gamma_2')(1-\frac{r_2}{r_3})+\gamma_2'(1-\frac{r_2+1}{r_3})}|D^{r_3}\phi|_0^{(1-\gamma_2')\frac{r_2}{r_3}+\gamma_2'\frac{r_2+1}{r_3}}$$

On the other hand, by proposition 6.5,

$$|D^{r_3}\phi|_0 \le 3|D^{r_3-1}\phi|_0^{\frac{\gamma_3'}{1+\gamma_3'}}|D^{r_3}\phi|_{\gamma_3'}^{\frac{1}{1+\gamma_3'}}$$

and we also have:

$$|D^{r_3-1}\phi|_0 \le C_{82}(r_3)|\phi|_0^{\frac{1}{r_3}}|D^{r_3}\phi|_0^{\frac{r_3-1}{r_3}}$$

Therefore,

$$|D^{r_3}\phi|_0^{1-(1-\frac{1}{r_3})\frac{\gamma_3'}{1+\gamma_3'}} \le 3(C_{82}(r_3))^{\frac{\gamma_3'}{1+\gamma_3'}}|\phi|_0^{\left(\frac{\gamma_3'}{1+\gamma_3'}\right)\frac{1}{r_3}}|D^{r_3}\phi|_{\gamma_3'}^{\frac{1}{1+\gamma_3'}}$$

Therefore,

$$|D^{r_3}\phi|_0 \le 3^{\frac{1+\gamma_3'}{1+\gamma_3'/r_3}}(C_{82}(r_3))^{\frac{\gamma_3'}{1+\gamma_3'/r_3}}|\phi|_0^{\left(\frac{\gamma_3'/r_3}{1+\gamma_3'/r_3}\right)\frac{1}{r_3}}|D^{r_3}\phi|_{\gamma_3'}^{\frac{1}{1+\gamma_3'/r_3}} \tag{74}$$

Therefore,

$$|D^{r_2}\phi|_{\gamma_2'} \le C_{82}(\gamma_3)|\phi|_0^{1-\frac{\gamma_2}{\gamma_3}}|D^{r_3}\phi|_{\gamma_3'}^{\frac{\gamma_2}{\gamma_3}}$$

with $C_{82}(\gamma_3) \le 3^2 C_{82}(r_3) \le 2^{(r_3+1)^2}$ if $r_3 \ge 2$ and $C_{82}(\gamma_3) \le 2^{(r_3+1)^2+1}$ if $r_3 = 1$.

Finally, if $r_2 = r_3 = r$ and $\gamma_2' < \gamma_3'$, then

$$|D^r\phi|_{\gamma_2'} \le 2|D^r\phi|_0^{1-\frac{\gamma_2'}{\gamma_3'}}|D^r\phi|_{\gamma_3'}^{\frac{\gamma_2'}{\gamma_3'}}$$

By applying estimation (74), we still have:

$$|D^{r_2}\phi|_{\gamma_2'} \le C_{82}(\gamma_3)|\phi|_0^{1-\frac{\gamma_2}{\gamma_3}}|D^{r_3}\phi|_{\gamma_3'}^{\frac{\gamma_2}{\gamma_3}}$$

with $C_{82}(\gamma_3) \le 2^{(r_3+1)^2}$ if $r_3 \ge 2$ and $C_{82}(\gamma_3) \le 2^{(r_3+1)^2+1}$ if $r_3 = 1$.
This completes the proof of proposition 5.3.

6.3 Proof of lemma 5.6.

If $\gamma < 1$, we prove the lemma directly. We prove lemma 5.6 for $\gamma \geq 1$. This is necessary to compute the constant. We write $\gamma = r + \gamma'$ with r integer and $0 \leq \gamma' < 1$. We need the Faa-di-Bruno formula. We first recall this formula and a related property (lemma 6.6). After that, we prove the lemma.

The Faa-di-Bruno formula states that for any integer $u \geq 1$ and functions f and g of class C^u,

$$D^u\left[f(g(x))\right] = \sum_{j=0}^{u} D^j f(g(x)) B_{u,j}\left(Dg(x), D^2 g(x), \dots, D^{(u-j+1)} g(x)\right)$$

where $B_{u,j}$ are the Bell polynomials, given by:

$$B_{u,j}(x_1, x_2, \dots, x_{u-j+1}) = \sum \frac{u!}{l_1! l_2! \cdots l_{u-j+1}!} \left(\frac{x_1}{1!}\right)^{l_1} \left(\frac{x_2}{2!}\right)^{l_2} \cdots \left(\frac{x_{u-j+1}}{(u-j+1)!}\right)^{l_{u-j+1}}$$

The sum extends over all sequences $l_1, l_2, l_3, ..., l_{u-j+1}$ of non-negative integers such that:$l_1 + l_2 + ... = j$ and $l_1 + 2l_2 + 3l_3 + ... = u$. The cardinal of this set of sequences is less than $\#\{(i_1, ..., i_l)/i_1 + i_2 + ... + i_l = l\} = \frac{(2l-1)!}{l!(l-1)!}$ (see page 51).

We obtain the estimation, for any x, and $u \geq 1$:

$$\left|B_{u,j}\left(Dg(x), D^2 g(x), \dots, D^{(u-j+1)} g(x)\right)\right| \leq \frac{(2u-1)!}{(u-1)!} \|Dg\|_{u-1}^{j} \tag{75}$$

We also need the lemma:

Lemma 6.6. *Let $u \geq 1$ and $u \geq j \geq 0$ be integers. Let $a_1, ..., a_{u-j+1}, x_1, ..., x_{u-j+1}$ be real numbers, let $a \geq \max\{|a_k|;\ 1 \leq k \leq u-j+1\}$, $x \geq \max\{|x_k| \vee 1;\ 1 \leq k \leq u-j+1\}$. Suppose $x + a \geq 1$. Let $B_{u,j}$ be a Bell polynomial. We have :*

$$|B_{u,j}(x_1 + a_1, ..., x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, ..., x_{u-j+1})| \leq u^2 \frac{(2u)!}{2} a(x+a)^u$$

Proof. Let $u + 1 \geq p \geq 1$ and $l_1, ..., l_p$ be integers. We have:

$$(x_1+a_1)^{l_1}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_p^{l_p} = \sum_{i=1}^{p} x_1^{l_1}...x_{i-1}^{l_{i-1}}(x_i+a_i)^{l_i}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_i^{l_i}(x_{i+1}+a_{i+1})^{l_{i+1}}...(x_p+a_p)^{l_p}$$

$$(x_1+a_1)^{l_1}...(x_p+a_p)^{l_p} - x_1^{l_1}...x_p^{l_p} = \sum_{i=1}^{p} x_1^{l_1}...x_{i-1}^{l_{i-1}}(x_{i+1}+a_{i+1})^{l_{i+1}}...(x_p+a_p)^{l_p}\left[(x_i+a_i)^{l_i} - x_i^{l_i}\right]$$

(with the conventions $x_1^{l_1}...x_0^{l_0} = 1$ and $x_{p+1}^{l_{p+1}}...x_p^{l_p} = 1$).

Since $|(x_i + a_i)^{l_i} - x_i^{l_i}| \leq l_i|a_i|(|x_i| + |a_i|)^{l_i-1} \leq l_i a(x+a)^{l_i-1}$, $l_i \leq u$ and $x + a \geq 1$, we obtain:

$$|B_{u,j}(x_1 + a_1, ..., x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, ..., x_{u-j+1})| \leq a(u-j+1)uB_{u,j}(x+a, ..., x+a)$$

The formula giving the Bell polynomials implies:

$$B_{u,j}(x+a, ..., x+a) \leq \frac{(2u)!}{2}(x+a)^u$$

□

For any $1 \leq u \leq r$,

$$|D^u(\phi(\psi(x))| \leq (u+1)\|D\phi\|_{r-1}\frac{(2u)!}{2}\|D\psi\|_{u-1}^u$$

We estimate $|D^r(\phi \circ \psi)|_{\gamma'}$. The Faa-di-Bruno formula applied to $\phi \circ \psi$ gives:

$$|D^r[\phi(\psi(x))] - D^r[\phi(\psi(y))]| \leq \sum_{j=0}^{r} |D^j\phi(\psi(x))|$$

$$|B_{r,j}\left(D\psi(x), \ldots, D^{(r-j+1)}\psi(x)\right) - B_{r,j}\left(D\psi(y), \ldots, D^{(r-j+1)}\psi(y)\right)| +$$

$$|D^j\phi(\psi(x)) - D^j\phi(\psi(y))||B_{r,j}\left(D\psi(y), \ldots, D^{(r-j+1)}\psi(y)\right)|$$

Moreover, for $1 \leq j \leq r$, $|D^j\phi(\psi(x))| \leq \|D\phi\|_{r-1}$, and

$$|D^j\phi(\psi(x)) - D^j\phi(\psi(y))| \leq |D^j\phi|_{\gamma'}|\psi(x) - \psi(y)|^{\gamma'} \leq \|D\phi\|_{\gamma-1}|D\psi|_0^{\gamma'}|x-y|^{\gamma'}$$

Estimation (75) gives:

$$\left|B_{r,j}\left(D\psi(y), D^2\psi(y), \ldots, D^{(r-j+1)}\psi(y)\right)\right| \leq \frac{(2r!)}{2}\|D\psi\|_{r-1}^r$$

We apply lemma 6.6: let $x_i = D^i\psi(x)$ and $a_i = D^i\psi(y) - D^i\psi(x)$. Let $x = \|D\psi\|_{r-1}$ and $a = \max_{1\leq i\leq r-j+1}|D^i\psi(x) - D^i\psi(y)|$. We have:

$$|B_{r,j}\left(D\psi(x), \ldots, D^{(r-j+1)}\psi(x)\right) - B_{r,j}\left(D\psi(y), \ldots, D^{(r-j+1)}\psi(y)\right)| \leq r^2\frac{(2r!)}{2}(2\|D\psi\|_{\gamma-1})^r \max_{1\leq i\leq r-j+1}|D^i\psi(x) - D^i\psi(y)|$$

$$|B_{r,j}\left(D\psi(x), \ldots, D^{(r-j+1)}\psi(x)\right) - B_{r,j}\left(D\psi(y), \ldots, D^{(r-j+1)}\psi(y)\right)| \leq r^2 2^r\frac{(2r!)}{2}\|D\psi\|_{\gamma-1}^{r+1}|x-y|^{\gamma'}$$

Therefore, we get:

$$|D^r(\phi \circ \psi)|_{\gamma'} \leq \|D\phi\|_{\gamma-1}\|D\psi\|_{\gamma-1}^{\lfloor\gamma\rfloor+1}\frac{(2r)!}{2}(r+1)(r^2 2^r + 1) \leq \|D\phi\|_{\gamma-1}\|D\psi\|_{\gamma-1}^{\lfloor\gamma\rfloor+1}((2r)!)^2$$

6.4 Estimates on some polynomials

Lemma 5.4 is used for some specific polynomials. There exist A_l, B_l, G_l^r, E_l^r, polynomials of l variables $X_1, ..., X_l$ homogeneous of weight l if X_i has weight i, such that, for $l \geq 1$, and for any diffeomorphisms g and h sufficiently differentiable, we have [14, p. 337-338]:

$$D^{l+1}g = A_l\left(D\log Dg, ..., D^l\log Dg\right)Dg \tag{76}$$

$$D^l\log Dg = B_l\left(\frac{D^2g}{Dg}, ..., \frac{D^{l+1}g}{Dg}\right) \tag{77}$$

For $r \geq 0$,

$$D^r \log D(g\circ h) = (D^r \log Dg\circ h)(Dh)^r + D^r \log Dh + \sum_{l=1}^{r-1} D^{r-l} \log Dg\circ h(Dh)^{r-l} G_l^r(D\log Dh, ..., D^l \log Dh) \tag{78}$$

For $r \geq 0$ and $n \geq 1$,

$$D^r \log Dg^n = \sum_{l=0}^{r-1} \sum_{i=0}^{n-1} (D^{r-l} \log Dg \circ g^i)(Dg^i)^{r-l} E_l^r(D\log Dg^i, ..., D^l \log Dg^i) \tag{79}$$

Lemma 5.4 uses the following estimate:

Claim 6.7. *Let $P = A_l, B_l, G_l^r$ or E_l^r. Write $P = \sum_{\sum_{k=1}^l kj_k = l} a_{j_i,...,j_l} X_1^{j_1} ... X_l^{j_l}$. We have:*

$$\sum_{\sum_{k=1}^l kj_k = l} |a_{j_i,...,j_l}| \leq (4(l+1))^{4(l+1)}$$

Proof. For example, if $P = B_l$, we have

$$B_{l+1}\left(\frac{D^2 g}{Dg}, ..., \frac{D^{l+2} g}{Dg}\right) = DB_l\left(\frac{D^2 g}{Dg}, ..., \frac{D^{l+1} g}{Dg}\right)$$

Since

$$D\left(\frac{D^u g}{Dg}\right) = \frac{D^{u+1} g}{Dg} - \frac{D^u g}{Dg}\frac{D^2 g}{Dg}$$

then

$$B_{l+1} = \sum_{i=1}^{l} (X_{i+1} - X_i X_1)\frac{\partial B_l}{\partial X_i}$$

Let $b_l = \max |a_{j_i,...,j_l}|$. We have: $b_{l+1} \leq 2l^2 b_l$. Thus $b_l \leq 2^l((l-1)!)^2$. Moreover, the sum $\sum_{\sum_{k=1}^l kj_k = l} |a_{j_i,...,j_l}|$ has less than $\frac{(2l-1)!}{l!(l-1)!}$ terms. Hence the claim for $P = B_l$. The computations are analogous for the other polynomials (and analogous to the proof of lemma 4.4 in appendix 6.1).

□

To obtain lemma 5.4, we apply estimations (47) and (48) to each monomial of P, and we apply this claim.

References

[1] VI Arnol'd. Small denominators I: Mappings of the circumference into itself. *American Mathematical Society Translations*, 46:213–284, 1965.

[2] M Benhenda. Circle Diffeomorphisms: Quasi-reducibility and Commuting Diffeomorphisms. *submitted*.

[3] R. De la Llave. A tutorial on KAM theory. In *Smooth Ergodic Theory and Its Applications: Proceedings of the AMS Summer Research Institute on Smooth Ergodic Theory and Its Applications, July 26-August 13, 1999, University of Washington, Seattle*, volume 69, page 175. Amer Mathematical Society, 1999.

[4] B. Fayad and K. Khanin. Smooth linearization of commuting circle diffeomorphisms. *Annals of Mathematics*, 170:961–980, 2009.

[5] M.R. Herman. Sur la Conjugaison Différentiable des Difféomorphismes du Cercle a des Rotations. *Publications Mathématiques de L'IHÉS*, 49(1):5–233, 1979.

[6] L. Hormander. The boundary problems of physical geodesy. *Archive for Rational Mechanics and Analysis*, 62(1):1–52, 1976.

[7] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54. Cambridge University Press, 1996.

[8] Y. Katznelson and D. Ornstein. The absolute continuity of the conjugation of certain diffeomorphisms of the circle. *Ergodic Theory Dynamical Systems*, 9(4):681–690, 1989.

[9] Y. Katznelson and D. Ornstein. The differentiability of the conjugation of certain diffeomorphisms of the circle. *Ergodic Theory Dynamical Systems*, 9(4):643–680, 1989.

[10] K. Khanin and A. Teplinsky. Herman's theory revisited. *Inventiones Mathematicae*, 178(2):333–344, 2009.

[11] KM Khanin and Y.G. Sinai. A new proof of M. Herman's theorem. *Communications in Mathematical Physics*, 112(1):89–101, 1987.

[12] J. Moser. A rapidly convergent iteration method and non-linear partial differential equations. i. *Ann. Scuola Norm. Sup. Pisa (3)*, 20:265–315, 1966.

[13] Y.G. Sinai and KM Khanin. Smoothness of conjugacies of diffeomorphisms of the circle with rotations. *Russian Mathematical Surveys*, 44(1):69–99, 1989.

[14] J.C. Yoccoz. Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. *Annales scientifiques de l'École Normale Supérieure Sér. 4*, 17(3):333–359, 1984.

[15] J.C. Yoccoz. Analytic linearization of circle diffeomorphisms. *Dynamical systems and small divisors*, pages 125–173, 2002.